Math 702 Problem Set 1

due Friday 2/14/2014

Problem 1. When the relation $\mathfrak{M}_1 \subset \mathfrak{M}_2$ holds between two σ-algebras on the same set, we say that $\mathfrak{M}_1$ is ***smaller*** (or ***coarser***) than $\mathfrak{M}_2$, and $\mathfrak{M}_2$ is ***larger*** (or ***finer***) than $\mathfrak{M}_1$.

(i) Given a map $f : X \to Y$ and a σ-algebra $\mathfrak{M}$ on X, verify that

$$f_* \mathfrak{M} = \{E \subset Y : f^{-1}(E) \in \mathfrak{M}\}$$

is the largest σ-algebra on Y which makes f measurable.

(ii) Given a map $f : X \to Y$ and a σ-algebra $\mathfrak{N}$ on Y, verify that

$$f^* \mathfrak{N} = \{f^{-1}(E) : E \in \mathfrak{N}\}$$

is the smallest σ-algebra on X which makes f measurable.

(iii) If $f : \mathbb{R}^2 \to \mathbb{R}$ is the projection $f(x, y) = x$, describe elements of $f^* \mathfrak{B}_{\mathbb{R}}$ and $f_* \mathfrak{B}_{\mathbb{R}^2}$. Here $\mathfrak{B}_{\mathbb{R}}$ and $\mathfrak{B}_{\mathbb{R}^2}$ are the Borel σ-algebras of $\mathbb{R}$ and $\mathbb{R}^2$.

Problem 2.

(i) If $f, g : X \to [-\infty, +\infty]$ are measurable, show that the sets

$$\{x \in X : f(x) < g(x)\} \qquad \text{and} \qquad \{x \in X : f(x) = g(x)\}$$

are measurable (Caution: Forming $f - g$ can be problematic.)

(ii) Suppose $f_n : X \to [-\infty, +\infty]$ is a sequence of measurable functions. Show that the set of points at which $\lim_{n \to \infty} f_n$ exists is measurable.

Problem 3. Consider the measurable space $(X, \mathfrak{M})$ in which X is an uncountable set and $\mathfrak{M}$ is the σ-algebra of all $E \subset X$ such that either E or E^c is countable. Describe measurable functions $X \to \mathbb{R}$.

Problem 4. (Borel-Cantelli Lemma) Let $(X, \mathfrak{M}, \mu)$ be a measure space and $\{E_n\}$ be a sequence in $\mathfrak{M}$ such that $\sum_n \mu(E_n) < +\infty$. Show that almost every point of X belongs to at most finitely many of the E_n. (Hint: Let A be the set of points in X that belong to infinitely many of the E_n. Use

$$A = \bigcap_{k \geq 1} \bigcup_{n \geq k} E_n$$

to prove that $\mu(A) = 0$.)

Problem 5. Recall that δ_p is the unit mass (=Dirac measure) at p.

(i) If $f : X \to Y$ is a map and $p \in X$, what is the push-forward $f_* \delta_p$?

(ii) Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $f : \mathbb{T} \to \mathbb{T}$ be the 120° rotation defined by $f(z) = e^{2\pi i/3}z$. What is the limit of the probability measures

$$\frac{1}{n}\sum_{i=0}^{n-1}(f^{\circ i})_*\delta_1$$

as $n \to \infty$? (As usual, $f^{\circ i}$ is the i-th iterate of f.)

Problem 6. Show that every infinite σ-algebra has uncountably many elements. (Hint: Suppose $\mathfrak{M}$ is a countably infinite σ-algebra on X, and for $x \in X$ let $A_x \in \mathfrak{M}$ be the intersection of all elements of $\mathfrak{M}$ that contain x. Show that $A_x \cap A_y \neq \emptyset$ implies $A_x = A_y$, so every element of $\mathfrak{M}$ is the disjoint union of a collection of A_x's. Use this to construct a bijection between $\mathfrak{M}$ and the set of all subsets of $\mathbb{N}$, which would give the desired contradiction.)