Math 702 Problem Set 3

due Friday 2/28/2014

Unless otherwise stated, $(X, \mathfrak{M}, \mu)$ is a given measure space.

Problem 1. Suppose $\mu(X) < +\infty$ and $f_n : X \to \mathbb{C}$ is a sequence of measurable functions. Show that there is a sequence $\{c_n\}$ of positive numbers such that $c_n f_n \to 0$ a.e. in X. (Hint: For each n there is a large enough $\alpha_n > 0$ such that $\mu(\{x \in X : |f_n(x)| > \alpha_n\}) < 1/2^n$.)

Problem 2. Let $f \in L^1(\mu)$. Show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |f| \, d\mu < \varepsilon$. (Hint: Proceed in one of the following ways: (1) Work with the sets $A_n = \{x \in X : |f(x)| > n\}$ and use the fact that $\int_{A_n} |f| \, d\mu \to 0$ as $n \to \infty$; or (2) Suppose there is an $\varepsilon > 0$ and sets E_n with $\mu(E_n) < 1/2^n$ such that $\int_{E_n} |f| \, d\mu > \varepsilon$ for all n. Look at the integral of $|f|$ over $\bigcup_{k \geq n} E_k$ and let $n \to \infty$ to reach a contradiction.)

Problem 3. *(Integrability as summability)* Let $f : X \to \mathbb{C}$ be measurable and define $A_n = \{x \in X : 2^n < |f(x)| \leq 2^{n+1}\}$ for $n \in \mathbb{Z}$. Show that

$$f \in L^1(\mu) \quad \text{if and only if} \quad \sum_{n=-\infty}^{\infty} 2^n \mu(A_n) < +\infty.$$

Problem 4. Suppose $f_n, f \in L^1(\mu)$, $f_n \to f$ a.e. on X, and $\int_X f_n \, d\mu \to \int_X f \, d\mu$. Prove that $\int_E f_n \, d\mu \to \int_E f \, d\mu$ for every measurable set $E \subset X$. (Hint: It suffices to consider the case where f_n, f are positive.)

Problem 5. *(Integrals depending on a real parameter)* Let I be an open interval in $\mathbb{R}$ and $f : X \times I \to \mathbb{R}$ be a function such that $f(\cdot, t) : X \to \mathbb{R}$ is integrable for each $t \in I$. Define $F : I \to \mathbb{R}$ by

$$F(t) = \int_X f(x, t) \, d\mu.$$

(i) Suppose $f(x, \cdot) : I \to \mathbb{R}$ is continuous for all $x \in X$ and there is a $g \in L^1(\mu)$ such that $|f(x, t)| \leq g(x)$ for all $(x, t) \in X \times I$. Show that F is continuous on I.

(ii) Suppose $f(x, \cdot) : I \to \mathbb{R}$ is differentiable for all $x \in X$ and there is a $g \in L^1(\mu)$ such that $|(\partial f/\partial t)(x, t)| \leq g(x)$ for all $(x, t) \in X \times I$. Show that F is differentiable on I and

$$\frac{dF}{dt} = \int_X \frac{\partial f}{\partial t}(x, t) \, d\mu \qquad (t \in I).$$

(Hint: For (ii), take any sequence $t_n \to t$ in I and apply the dominated convergence theorem to the sequence of measurable functions $x \mapsto (f(x, t_n) - f(x, t))/(t_n - t)$.)

Problem 6. Let $\mu(X) < +\infty$. The ***average value*** of $f \in L^1(\mu)$ over a measurable set E with $\mu(E) > 0$ is defined by

$$f_E = \frac{1}{\mu(E)} \int_E f \, d\mu.$$

Let $T \subset \mathbb{C}$ be non-empty and closed.

(i) If $f_E \in T$ for every E, show that $f(x) \in T$ for a.e. $x \in X$.

(ii) Show by an example that $f(x) \in T$ for a.e. $x \in X$ does not imply $f_E \in T$ for every E.

(iii) (Bonus) What geometric property of T will guarantee the implication

$$f(x) \in T \text{ for a.e. } x \in X \implies f_E \in T \text{ for every } E?$$

Prove your claim.

(Hint for (i): Express the open set $\mathbb{C} \setminus T$ as a countable union of closed disks D_n and show $f^{-1}(D_n)$ has measure zero for all n.)