# Math 702 Problem Set 5

**due Friday 3/14/2014**

Throughout $m$ denotes Lebesgue measure on $\mathbb{R}^d$. The space $L^1(m)$ of Lebesgue integrable functions on $\mathbb{R}^d$ is denoted more traditionally by $L^1(\mathbb{R}^d)$.

**Problem 1.** Find, with justification, the limits

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx \qquad \text{and} \qquad \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{n\sin(x/n)}{x(x^2+1)}\,dx.$$

**Problem 2.** Consider the function $g(x)=\sum_{n=1}^{\infty}2^{-n}f(x-r_n)$, where

$$f(x)=\begin{cases} x^{-1/2} & \text{if } x\in(0,1) \\ 0 & \text{if } x\notin(0,1). \end{cases}$$

and $\{r_n\}$ is any enumeration of $\mathbb{Q}$. Show that $g\in L^1(\mathbb{R})$ so $g<+\infty$ a.e., but $g$ is unbounded on every interval.

**Problem 3.**

(i) For each $0<\varepsilon<1$ construct an open dense set $U\subset[0,1]$ such that $m(U)=\varepsilon$.

(ii) Construct a Borel set $E\subset[0,1]$ such that

$$0<m(E\cap I)<m(I)$$

for every interval $I\subset[0,1]$.

(Hint: For (i), think of fat Cantor sets. For (ii), construct $E$ as the intersection of a decreasing sequence of suitably chosen open dense sets.)

**Problem 4.**

(i) Let $E\subset\mathbb{R}^d$. Show that $m(E)=0$ iff for every $\varepsilon>0$ there are countably many open (Euclidean) balls $\{B(x_n,r_n)\}$ such that $E\subset\bigcup B(x_n,r_n)$ and $\sum r_n^d<\varepsilon$.

(ii) Suppose $f:\mathbb{R}^d\to\mathbb{R}^d$ is a Lipschitz map, i.e., there is an $M>0$ such that

$$\|f(x)-f(y)\|\le M\|x-y\| \qquad \text{for all } x,y\in\mathbb{R}^d.$$

Show that $m(f(E))=0$ whenever $m(E)=0$.

**Problem 5.** Suppose $f:\mathbb{R}^d\to\mathbb{R}$ is Lebesgue measurable. Show that there are Borel maps $g,h:\mathbb{R}^d\to\mathbb{R}$ such that $g(x)\le f(x)\le h(x)$ for every $x\in\mathbb{R}^d$ and $g(x)=f(x)=h(x)$ for a.e. $x\in\mathbb{R}^d$. (Hint: It suffices to assume $f\ge 0$. Treat the case of positive simple functions first.)

**Problem 6.** Give an example of a translation-invariant Borel measure on $\mathbb{R}^d$ that is not a multiple of Lebesgue measure $m$. Why doesn't your example violate the uniqueness theorem that we proved for $m$?

**Problem 7.** (Bonus) The ***support*** of a Borel measure $\mu$ on a metric space $X$ is defined by

$$\operatorname{supp}(\mu) = \{x \in X : \mu(B(x, r)) > 0 \text{ for every } r > 0\}.$$

(i) If $X$ is separable, show that $\operatorname{supp}(\mu)$ is the smallest closed set in $X$ whose complement has measure zero. In particular, $\operatorname{supp}(\mu)$ is non-empty (unless $\mu = 0$).

(ii) Show that every non-empty closed set in $\mathbb{R}^d$ is the support of a Radon measure.

(Hint for (ii): Given a non-empty closed set in $\mathbb{R}^d$, choose a countable dense subset of it, use this subset to define a positive linear functional on $\mathscr{C}_c(\mathbb{R}^d)$, and apply the Riesz representation theorem.)