Math 702 Problem Set 6

due Friday 3/21/2014

Measurable sets and functions in Euclidean spaces are always meant to be Lebesgue measurable. A ***full measure*** set is one whose complement has measure zero.

Problem 1.

(i) If $f : \mathbb{R} \to \mathbb{R}$ is measurable and $\varepsilon > 0$, prove that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $m\big(\{x : f(x) \neq g(x)\}\big) \leq \varepsilon$.

(ii) Show by an example that there may not be such a g for $\varepsilon = 0$.

Problem 2. Suppose $f_n : [0,1] \to \mathbb{R}$ is a sequence of measurable functions such that $f_n \to 0$ a.e. in $[0,1]$.

(i) Show that there is a subsequence $\{f_{n_k}\}$ for which the series $\sum_{k=1}^{\infty} |f_{n_k}|$ converges a.e. in $[0,1]$.

(ii) Can we find a subsequence $\{f_{n_k}\}$ for which $\sum_{k=1}^{\infty} |f_{n_k}|$ converges uniformly on a full measure subset of $[0,1]$?

(Hint: Use Egoroff's theorem.)

Problem 3. Give an example of a monotone class $\mathfrak{C}$ in $\mathbb{R}$ such that

(i) $\mathfrak{C}$ is not a σ-algebra, and
(ii) $E \in \mathfrak{C}$ implies $\mathbb{R} \smallsetminus E \in \mathfrak{C}$, and
(iii) $\mathfrak{C}$ has infinitely many elements.

Problem 4. Recall that $\mathfrak{B}_X$ is the Borel σ-algebra of the metric space X.

(i) Show that for any pair of metric spaces X and Y,

$$\mathfrak{B}_X \otimes \mathfrak{B}_Y \subset \mathfrak{B}_{X \times Y}.$$

(ii) If X is separable, prove that

$$\mathfrak{B}_X \otimes \mathfrak{B}_Y = \mathfrak{B}_{X \times Y}.$$

As a special case, for $\mathfrak{B}_d = \mathfrak{B}_{\mathbb{R}^d}$ we have the relation $\mathfrak{B}_d \otimes \mathfrak{B}_k = \mathfrak{B}_{d+k}$ once we identify the Cartesian product $\mathbb{R}^d \times \mathbb{R}^k$ with $\mathbb{R}^{d+k}$.

(Hint: For (i), show that the product of two Borel sets is Borel. For (ii), show that every open set in $X \times Y$ is a countable union of open rectangles.)

Problem 5. Investigate integrability of the following functions on the given product space:

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \qquad \text{on } [0,1]\times[0,1]$$
$$g(x,y) = \frac{xy}{(x^2+y^2)^2} \qquad \text{on } [-1,1]\times[-1,1].$$

Problem 6. Suppose f is integrable on [0, 1]. Show that the function $g(x) = \int_x^1 t^{-1} f(t)\,dt$ is integrable on [0, 1] and $\int_0^1 g = \int_0^1 f$.

Problem 7. (Bonus) Does there exist a full measure subset of $\mathbb{R}^2$ which contains no measurable rectangle $A \times A$ of positive measure?