Math 702 Problem Set 8 Solutions

In the following problems, \((X, \mu)\) is always a measure space,

\[
\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p} \quad (0 < p < \infty),
\]

and \(L^p = L^p(\mu)\) is the space of all measurable \(f : X \to \mathbb{C}\) for which \(\|f\|_p < \infty\).

**Problem 1.** Let \(f : X \to \mathbb{C}\) be measurable. Prove the following statements:

(i) If \(0 < r < s \leq \infty\) and \(f \in L^r \cap L^s\), then \(f \in L^p\). In other words, the set \(\{p \in (0, \infty) : f \in L^p\}\) is connected.

First suppose \(s < \infty\) and set \(A = \{x : |f(x)| < 1\}\). Then

\[
\|f\|_p^p = \int_X |f|^p \, d\mu = \int_A |f|^p \, d\mu + \int_{A^c} |f|^p \, d\mu
\]

\[
\leq \int_A |f|^r \, d\mu + \int_{A^c} |f|^s \, d\mu \leq \|f\|_r^r + \|f\|_s^s,
\]

which shows \(f \in L^p\). If \(s = \infty\), the inequality

\[
(1)
\]

\[
\|f\|_p^p = \int_X |f|^p \, d\mu = \int_X |f|^{p-r} |f|^r \, d\mu \leq \|f\|_\infty^{p-r} \|f\|_r^r
\]

shows, again, that \(f \in L^p\).

(ii) If \(0 < r < s \leq \infty\) and \(f \in L^r \cap L^s\), and if \(f\) is not a.e. equal to \(0\), the function \(\varphi(p) = \log \|f\|_p^p\) is convex in the interval \((r, s)\).

By part (i), \(\varphi\) is finite-valued in \((r, s)\). Take \(x, y \in (r, s)\) and \(0 < t < 1\). Apply Hölder’s inequality for the pair of conjugate exponents \(p = 1/t, q = 1/(1-t)\) to get

\[
\varphi((1-t)x + ty) = \log \left(\int |f|^{(1-t)x+ty} \right) = \log \left(\int |f|^{(1-t)x} \cdot |f|^ty \right)
\]

\[
\leq \log \left[\left(\int |f|^x \right)^{(1-t)} \left(\int |f|^y \right)^t\right]
\]

\[
= (1-t) \log \left(\int |f|^x \right) + t \log \left(\int |f|^y \right)
\]

\[
= (1-t)\varphi(x) + t\varphi(y).
\]

This proves convexity of \(\varphi\).

**Problem 2.** Suppose \(f : X \to \mathbb{C}\) is measurable and \(f \in L^r\) for some \(0 < r < \infty\). Prove the following statements about the behavior of \(\|f\|_p\) for large and small \(p\):
(i) \( \lim_{p \to \infty} \|f\|_p = \|f\|_\infty \) whether or not \( \|f\|_\infty \) is finite.

The result is trivial if \( \|f\|_\infty = 0 \) since in this case \( f = 0 \) a.e. so \( \|f\|_p = 0 \) for all \( p \). So let us assume \( \|f\|_\infty > 0 \). Let \( 0 < \lambda < \|f\|_\infty \) and consider the set \( A = \{ x : |f(x)| > \lambda \} \). Then \( \mu(A) > 0 \) by the definition of \( \|f\|_\infty \) and \( \mu(A) < \infty \) since \( f \in L^r \). Hence,

\[
\|f\|_p = \left( \int_X |f|^p \right)^{1/p} \geq \left( \int_A |f|^p \right)^{1/p} \geq \lambda \mu(A)^{1/p},
\]

which implies \( \liminf_{p \to \infty} \|f\|_p \geq \lambda \). Letting \( \lambda \to \|f\|_\infty \), we obtain

\[
(2) \liminf_{p \to \infty} \|f\|_p \geq \|f\|_\infty.
\]

This completes the proof if \( \|f\|_\infty = \infty \), so assume for the remainder of the proof that \( f \in L^\infty \). In this case, the estimate (1) in the solution of problem 1(i) shows that for every \( p > r \),

\[
\|f\|_p \leq \|f\|_r^{p/r} \|f\|_\infty^{1-r/p},
\]

which gives

\[
(3) \limsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty.
\]

The result now follows by combining (2) and (3).

(ii) If \( \mu(X) = 1 \), then

\[
\lim_{p \to 0} \|f\|_p = \exp \left( \int_X \log |f| \, d\mu \right)
\]

if we define \( \exp(-\infty) = 0 \).

Again, we may assume that \( f \) is not a.e. 0 since the result trivially holds in that case. First notice that \( f \in L^p \) if \( 0 < p < r \). In fact, by Hölder’s inequality for the pair of conjugate exponents \( \alpha = r/p \) and \( \beta = r/(r-p) \),

\[
\int |f|^p \leq \left( \int |f|^{p\alpha} \right)^{1/\alpha} \left( \int 1^\beta \right)^{1/\beta} = \|f\|_r^p,
\]

so \( \|f\|_p \leq \|f\|_r \). The same argument proves that \( \|f\|_p \) decreases as \( p \to 0 \), so \( \lim_{p \to 0} \|f\|_p \) exists.

Now Jensen’s inequality for the convex function \( -\log x \) shows that for \( 0 < p < r \),

\[
- \log \left( \int |f|^p \right) \leq - \int \log |f|^p.
\]
Rearranging, we obtain $\log \|f\|_p \geq \int \log |f|$ or $\|f\|_p \geq \exp(\int \log |f|)$. Letting $p \to 0$ then gives

$$\lim_{p \to 0} \|f\|_p \geq \exp\left(\int \log |f|\right).$$

For the reverse, use the inequality $\log x \leq x - 1$ for $x > 0$ to write

$$\log \|f\|_p^p \leq \|f\|_p^p - 1 = \int |f|^p - 1 = \int (|f|^p - 1),$$

where the last equality holds because $\mu(X) = 1$. This gives

$$\|f\|_p \leq \exp\left(\int \frac{|f|^p - 1}{p}\right).$$

On the set $A = \{x : |f| \geq 1\}$ the integrable functions $(|f|^p - 1)/p$ are positive and decrease to $\log |f|$ as $p \to 0$. Hence, by the monotone convergence theorem,

$$\lim_{p \to 0} \int_A \frac{|f|^p - 1}{p} = \int_A \log |f|.$$  

On the complement $A^c$, the functions $(1 - |f|^p)/p$ are positive and increase to $-\log |f|$ as $p \to 0$. Again, by the monotone convergence theorem,

$$\lim_{p \to 0} \int_{A^c} \frac{|f|^p - 1}{p} = \int_{A^c} \log |f|$$

(the latter integral might be $-\infty$). Adding the last two equations and using (5) gives

$$\lim_{p \to 0} \|f\|_p \leq \exp\left(\int \log |f|\right),$$

and the result follows by combining (4) and (6).

**Comment.** Here is an amusing consequence of (ii) when $X = \{1, \ldots, n\}$ and $\mu$ is the uniform probability measure $\mu(\{i\}) = 1/n$. Every function $X \to \mathbb{C}$ can be identified with the $n$-tuple $(x_1, \ldots, x_n)$ of its values on $X$, so $L^p(\mu)$ is just $\mathbb{C}^n$ equipped with the norm $\|x\|_p = ((1/n) \sum |x_i|^p)^{1/p}$. The statement (ii) then shows that

$$\lim_{p \to 0} \left(\frac{1}{n} \sum_{i=1}^n |x_i|^p\right)^{1/p} = \exp\left(\frac{1}{n} \sum_{i=1}^n \log |x_i|\right) = \left(\prod_{i=1}^n |x_i|\right)^{1/n}.$$

**Problem 3.** Suppose $\mu(X) < \infty$, $f \in L^\infty(\mu)$, and $\|f\|_\infty > 0$. Define

$$\alpha_n = \int_X |f|^n \, d\mu, \quad n = 1, 2, 3, \ldots$$

Show that

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$
The assumption $\|f\|_\infty > 0$ implies that $\alpha_n > 0$ for every $n$. Since

$$\alpha_{n+1} = \int |f|^{n+1} = \int |f| \cdot |f|^n \leq \|f\|_\infty \int |f|^n = \|f\|_\infty \alpha_n,$$

we have

(7) \quad \limsup_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty.

On the other hand, Hölder’s inequality for the pair of conjugate exponents $p = (n+1)/n$ and $q = n+1$ shows that

$$\int |f|^n \leq \left( \int |f|^{n+1} \right)^{n/(n+1)} \left( \int 1^{n+1} \right)^{1/(n+1)}$$

which gives

$$\alpha_{n+1}^n \leq \alpha_{n+1}^n \mu(X) \quad \text{or} \quad \frac{\alpha_{n+1}}{\alpha_n} \geq \alpha_{n+1}^{1/n} \mu(X)^{-1/n}.$$  

By problem 2(i), $\alpha_{n+1}^{1/n} = \|f\|_n \to \|f\|_\infty$ as $n \to \infty$. This shows

(8) \quad \liminf_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} \geq \|f\|_\infty,$

and the result follows from (7) and (8).

**Problem 4.** For each $0 < p \leq \infty$, find a function in $L^p(\mathbb{R})$ that does not belong to $L^r(\mathbb{R})$ for any $r \neq p$.

If $p = \infty$, the constant function 1 will do. For the remaining cases, it suffices to find an $f \in L^1(\mathbb{R}) \setminus \bigcup_{r \neq 1} L^r(\mathbb{R})$. For an arbitrary $0 < p < \infty$, the function $|f|^{1/p}$ will be in $L^p(\mathbb{R}) \setminus \bigcup_{r \neq p} L^r(\mathbb{R})$.

Finding such $f$ is easy. For example, let $f(x) = x^{-1} (\log x)^{-2}$ if $0 < x < 1/2$ or $x > 2$, and $f(x) = 0$ otherwise. Using the fact that $x^{-1} (\log x)^{-2}$ is the derivative of $- (\log x)^{-1}$, it is easily seen that $f \in L^1(\mathbb{R})$. Since $\log x$ is dominated by any positive power of $x$ as $x \to \infty$, we see that $f^r(x) \geq x^{-1}$ for all sufficiently small (resp. large) $x > 0$ if $r > 1$ (resp. $0 < r < 1$). It follows that $f \notin L^r(\mathbb{R})$ if $r \neq 1$.

**Problem 5.** The *essential range* of $f \in L^\infty(\mu)$ is the set $R_f$ of points $z \in \mathbb{C}$ for which $\mu(\{x \in X : |f(x) - z| < \varepsilon\})$ is positive for every $\varepsilon > 0$. Show that $R_f$ is compact. What relation can you establish between $R_f$ and $\|f\|_\infty$?

Since the set $\{x : |f(x)| > \|f\|_\infty\}$ has measure zero, it is clear that $R_f$ is contained in the closed disk $\{z \in \mathbb{C} : |z| \leq \|f\|_\infty\}$. To prove compactness of $R_f$, it is therefore enough to show that $R_f$ is closed. Suppose $z \notin R_f$ and find $\varepsilon > 0$ such that $f^{-1}(\mathbb{D}(z, \varepsilon))$ has measure zero. If $w \in \mathbb{D}(z, \varepsilon)$ and $r > 0$ is small enough to guarantee $\mathbb{D}(w, r) \subset \mathbb{D}(z, \varepsilon)$, then $f^{-1}(\mathbb{D}(w, r))$ too has measure zero, so $w \notin R_f$. This shows $\mathbb{D}(z, \varepsilon) \subset \mathbb{C} \setminus R_f$ and proves $\mathbb{C} \setminus R_f$ is open.
For the second question, we show that \( \| f \|_\infty \) is the outer radius \( r_{\text{out}} \) of the set \( R_f \):
\[
\| f \|_\infty = r_{\text{out}} = \max\{|z| : z \in R_f\}
\]
(the maximum is achieved since \( R_f \) is compact). In fact, \( r_{\text{out}} \leq \| f \|_\infty \) by what we observed above. Assume by way of contradiction that \( r_{\text{out}} < \| f \|_\infty \). The definition of \( \| f \|_\infty \) shows that \( \mu(f^{-1}(A)) > 0 \), where \( A = \{z \in \mathbb{C} : r_{\text{out}} < |z| \leq \| f \|_\infty\} \). Since \( A \cap R_f = \emptyset \), for every \( z \in A \) there is an \( \epsilon > 0 \) such that \( f^{-1}(\mathbb{D}(z, \epsilon)) \) has measure zero. Since \( A \) is covered by countably many such disks, this would imply \( f^{-1}(A) \) having measure zero, a contradiction.

**Problem 6.** A sequence \( f_n : X \to \mathbb{C} \) of measurable functions is said to converge in measure to a measurable function \( f : X \to \mathbb{C} \) if for every \( \epsilon > 0 \),
\[
\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \to 0 \quad \text{as} \quad n \to \infty.
\]
Prove the following statements:

(i) If \( \mu(X) < \infty \) and \( f_n \to f \) a.e., then \( f_n \to f \) in measure.

Take any \( \epsilon, \eta > 0 \). By Egoroff’s theorem, there is a measurable set \( E \subseteq X \) with \( \mu(X \setminus E) < \eta \) such that \( f_n \to f \) uniformly on \( E \). Find \( N \geq 1 \) such that \( x \in E \) and \( n > N \) imply \( |f_n(x) - f(x)| < \epsilon \). Then, if \( n > N \), the set \( \{x : |f_n(x) - f(x)| > \epsilon\} \) is contained in \( X \setminus E \) and therefore has measure \( < \eta \). This proves \( f_n \to f \) in measure.

(ii) If \( f_n \to f \) in \( L^p(\mu) \) for some \( 1 \leq p \leq \infty \), then \( f_n \to f \) in measure.

The case \( p = \infty \) is trivial, so let us assume \( 1 \leq p < \infty \). Given \( \epsilon > 0 \), let \( E_n = \{x : |f_n(x) - f(x)| > \epsilon\} \). Then
\[
\|f_n - f\|_p^p = \int_X |f_n - f|^p d\mu \geq \int_{E_n} |f_n - f|^p d\mu \geq \epsilon^p \mu(E_n),
\]
so
\[
0 \leq \mu(E_n) \leq \epsilon^{-p} \|f_n - f\|_p^p.
\]
Since \( \|f_n - f\|_p \to 0 \) by the assumption, it follows that \( \mu(E_n) \to 0 \).

(iii) If \( f_n \to f \) in measure, some subsequence of \( f_n \) converges to \( f \) a.e.

Since \( f_n \to f \) in measure, we can find a sequence \( 1 < n_1 < n_2 < n_3 < \cdots \) of integers such that
\[
E_k = \{x : |f_{n_k}(x) - f(x)| > \frac{1}{k} \}
\]
satisfies \( \mu(E_k) < \frac{1}{k^2} \) for all \( k \geq 1 \). Since \( \sum \mu(E_k) < \infty \), the Borel-Cantelli lemma shows that a.e. \( x \in X \) belongs to at most finitely many of the \( E_k \). In other words, for a.e. \( x \in X \) there is an integer \( N \geq 1 \) (depending on \( x \)) such that \( |f_{n_k}(x) - f(x)| \leq 1/k \) if \( k > N \). This means \( f_{n_k} \to f \) a.e. in \( X \).
(iv) Show that the converses of (i) and (ii) do not hold, so convergence in measure is weaker than both a.e. pointwise convergence and $L^p$ convergence.

The sequence $f_n = \chi_{J_n}$, where $\{J_n\}$ is an enumeration of the dyadic intervals in $[0, 1]$ clearly converges to $f = 0$ in measure, but for every $x \in [0, 1],$
$$\liminf_{n \to \infty} f_n(x) = 0 < 1 \leq \limsup_{n \to \infty} f_n(x),$$
so \{f_n\} does not converge anywhere in $[0, 1]$.

The sequence $f_n = n\chi_{[0,1/n]}$ converges to $f = 0$ in measure, but neither $\|f_n\|_p = n^{(p-1)/p}$ nor $\|f_n\|_\infty = n$ tends to 0 as $n \to \infty$.

Comment. By combining parts (ii) and (iii), we obtain another proof for the statement that every convergent sequence in $L^p(\mu)$ has an a.e. pointwise convergent subsequence (we previously obtained this as a byproduct of our proof of the completeness of $L^p(\mu)$).