CHAPTER 1

The Measurable Riemann Mapping Theorem

§1.1. Conformal structures on Riemann surfaces

Throughout, "smooth" will always mean C^∞. All surfaces are assumed to be smooth, oriented and without boundary. All diffeomorphisms are assumed to be smooth and orientation-preserving.

It will be convenient for our purposes to do local computations involving metrics in the complex-variable notation. Let X be a Riemann surface and $z = x + iy$ be a holomorphic local coordinate on X. The pair (x, y) can be thought of as local coordinates for the underlying smooth surface. In these coordinates, a smooth Riemannian metric λ has the local form

$$E\,dx^2 + 2F\,dx\,dy + G\,dy^2,$$

where E, F, G are smooth functions of (x, y) satisfying $E > 0$, $G > 0$ and $EG - F^2 > 0$. The associated inner product on each tangent space is given by

$$\begin{aligned}\left\langle a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\ ,\ c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}\right\rangle &= Eac + F(ad + bc) + Gbd \\ &= \begin{bmatrix} a & b \end{bmatrix}\ \Gamma\ \begin{bmatrix} c \\ d \end{bmatrix},\end{aligned}$$

where the symmetric positive definite matrix

$$\Gamma = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{1.1}$$

represents λ in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. In particular, the length of a tangent vector is given by

$$\left\| a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right\| = \sqrt{Ea^2 + 2Fab + Gb^2}.$$

Define two local sections of the complexified cotangent bundle $T^*X \otimes \mathbb{C}$ by

$$\begin{aligned} dz &:= dx + i\,dy \\ d\bar{z} &:= dx - i\,dy. \end{aligned}$$

These form a basis for each complexified cotangent space. The local sections

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$$

of the complexified tangent bundle $TX \otimes \mathbb{C}$ will form the dual basis at each point. The inner product on TX extends canonically to a Hermitian product on $TX \otimes \mathbb{C}$. The matrix of this Hermitian product in the basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}\}$ is given by

$$\hat\Gamma = P\,\Gamma\,P^*, \tag{1.2}$$

where

$$P = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}. \tag{1.3}$$

It follows from (1.1), (1.2) and (1.3) that

$$\hat\Gamma = \frac{1}{4}\begin{bmatrix} E+G & E-G-2iF \\ E-G+2iF & E+G \end{bmatrix}.$$

Let us introduce the quantities

$$\gamma := \frac{1}{2}(E+G+\sqrt{EG-F^2})^{1/2} \tag{1.4}$$

$$\mu := \frac{1}{4\gamma^2}(E-G+2iF) = \frac{E-G+2iF}{E+G+\sqrt{EG-F^2}}\,. \tag{1.5}$$

Note that

$$\gamma > 0 \quad \text{and} \quad |\mu| = \left(\frac{E+G-2\sqrt{EG-F^2}}{E+G+2\sqrt{EG-F^2}}\right)^{1/2} < 1. \tag{1.6}$$

A straightforward computation then shows that

$$\hat\Gamma = \frac{1}{2}\gamma^2\begin{bmatrix} 1+|\mu|^2 & 2\bar\mu \\ 2\mu & 1+|\mu|^2 \end{bmatrix}.$$

Since the Hermitian product on $TX \otimes \mathbb{C}$ is given by

$$\left\langle A\frac{\partial}{\partial z} + B\frac{\partial}{\partial \bar z}\,,\; C\frac{\partial}{\partial z} + D\frac{\partial}{\partial \bar z}\right\rangle = \begin{bmatrix} A & B \end{bmatrix}\,\hat\Gamma\,\begin{bmatrix} \bar C \\ \bar D \end{bmatrix},$$

it follows that

$$\left\| A\frac{\partial}{\partial z} + B\frac{\partial}{\partial \bar z}\right\|^2 = \frac{1}{2}\gamma^2\Big((1+|\mu|^2)(|A|^2+|B|^2) + 2\mu\,\bar A\,B + 2\bar\mu\,A\,\bar B\Big). \tag{1.7}$$

Now a tangent vector $A\frac{\partial}{\partial z} + B\frac{\partial}{\partial \bar{z}} \in TX \otimes \mathbb{C}$ is real (i.e., belongs to TX) if and only if $B = \bar{A}$. This simply follows from

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} = (a+ib)\frac{\partial}{\partial z} + (a-ib)\frac{\partial}{\partial \bar{z}}.$$

Thus, for real tangent vectors, the formula (1.7) reduces to

$$\left\| A\,\frac{\partial}{\partial z} + \bar{A}\,\frac{\partial}{\partial \bar{z}} \right\| = \gamma\,|A + \mu\,\bar{A}|.$$

The last expression suggests that if we are only concerned about *lengths* of *real* tangent vectors, the metric λ in the complex basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$ can be represented as

$$\lambda = \gamma(z)\,|dz + \mu(z)\,d\bar{z}|, \tag{1.8}$$

with γ and μ defined by (1.4) and (1.5).

Let us see how the quantities γ and μ associated with λ transform under a holomorphic change of coordinates $z \mapsto w$ on X:

$$\begin{aligned}
\gamma(z)\,|dz + \mu(z)\,d\bar{z}| &= \gamma(w)\,|dw + \mu(w)\,d\bar{w}| \\
&= \gamma(w(z))\,|w'(z)\,dz + \mu(w(z))\overline{w'(z)}\,d\bar{z}| \\
&= \gamma(w(z))\,|w'(z)| \left| dz + \mu(w(z))\frac{\overline{w'(z)}}{w'(z)}\,d\bar{z} \right|,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\gamma(z) &= \gamma(w(z))\,|w'(z)| \\
\mu(z) &= \mu(w(z))\,\frac{\overline{w'(z)}}{w'(z)}
\end{aligned}$$

or simply

$$\begin{aligned}
\gamma(z)\,|dz| &= \gamma(w)\,|dw| \\
\mu(z)\,\frac{d\bar{z}}{dz} &= \mu(w)\,\frac{d\bar{w}}{dw}.
\end{aligned} \tag{1.9}$$

It follows that $\gamma(z)\,|dz|$ and $\mu(z)\,\frac{d\bar{z}}{dz}$ are well-defined forms on X. Note that $z \mapsto |\mu(z)|$ is a well-defined function on X, even though $z \mapsto \mu(z)$ is not.

DEFINITION 1.1. $\mu = \mu(z)\,\frac{d\bar{z}}{dz}$ is called the ***Beltrami form*** associated with the metric λ. We say λ is a ***conformal metric*** if its Beltrami form is identically zero, in which case $\lambda = \gamma(z)\,|dz|$ is a positive multiple of the Euclidean metric in every local coordinate.

Conformal metrics exist on every Riemann surface. For example, the spherical metric $|dz|/(1+|z|^2)$ on $\hat{\mathbb{C}}$, the Euclidean metric $|dz|$ on $\mathbb{C}$, $\mathbb{C}^*$ and the tori $\mathbb{C}/\Lambda$, and the Poincaré metric on hyperbolic Riemann surfaces are all conformal metrics.

It is clear from (1.8) that two smooth metrics λ and $\hat{\lambda}$ have the same Beltrami forms if and only if they belong to the same ***conformal class***, which means $\hat{\lambda}/\lambda$ is a smooth positive function $X \to \mathbb{R}$. Each conformal class is also called a smooth ***conformal structure*** on X. On the other hand, given a Beltrami form $\mu = \mu(z)\,\frac{d\bar{z}}{dz}$ on X, we can pair it with an arbitrary conformal metric $\gamma(z)\,|dz|$ to construct the metric $\lambda = \gamma(z)\,|dz + \mu(z)\,d\bar{z}|$ with the Beltrami form μ. The ***canonical conformal structure*** of X is the one represented by any conformal metric $\gamma(z)|dz|$ on X. It corresponds to the ***zero Beltrami form*** μ_0 which vanishes identically in every local coordinate on X.

COROLLARY 1.2. *There is a one-to-one correspondence between smooth conformal structures on X and smooth Beltrami forms $\mu = \mu(z)\,\frac{d\bar{z}}{dz}$ which satisfy $|\mu(z)| < 1$ in every local coordinate z on X.*

Here is a more geometric description for a Beltrami form μ as a field of concentric ellipses on the tangent bundle of X. Fix a local coordinate $z = x + iy \cong (x, y)$ near a point $p \in X$ and let $\mu = \mu(z)\,\frac{d\bar{z}}{dz}$ in this coordinate. Consider the family of "circles" $E(p) = \{\mathbf{v} \in T_pX : \|\mathbf{v}\| = \text{const.}\}$ which depends only on μ and not on the choice of the representative metric. If $\mathbf{v} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} = (a+ib)\frac{\partial}{\partial z} + (a-ib)\frac{\partial}{\partial \bar{z}}$, then the "circles" $\|\mathbf{v}\| = \text{const.}$ in T_pX correspond to the loci $|(a+ib) + \mu(a-ib)| = \text{const.}$ in the real (a,b)-plane. Setting $\mu := re^{i\theta}$ and $\zeta := (a+ib)e^{-i\theta/2}$, we obtain the loci $|\zeta + r\bar{\zeta}| = \text{const.}$ in the ζ-plane, which is just the family of concentric ellipses with the minor axis along the real direction and the major axis along the imaginary direction, and with the ratio of the major to minor axis equal to $(1+r)/(1-r)$ (compare Fig. 1.1). Transferring this family back to the (a,b)-plane, it follows that $E(p)$ is a family of concentric ellipses in T_pX with

$$\text{angle of elevation of the minor axis} = \frac{1}{2}\arg\mu$$
$$\text{ratio of the major to minor axis} = \frac{1+|\mu|}{1-|\mu|}.$$

Note that in this geometric description, the zero Beltrami form μ_0 corresponds to the field of round circles $|\zeta|^2 = a^2 + b^2 = \text{const.}$ in the (a,b)-plane.

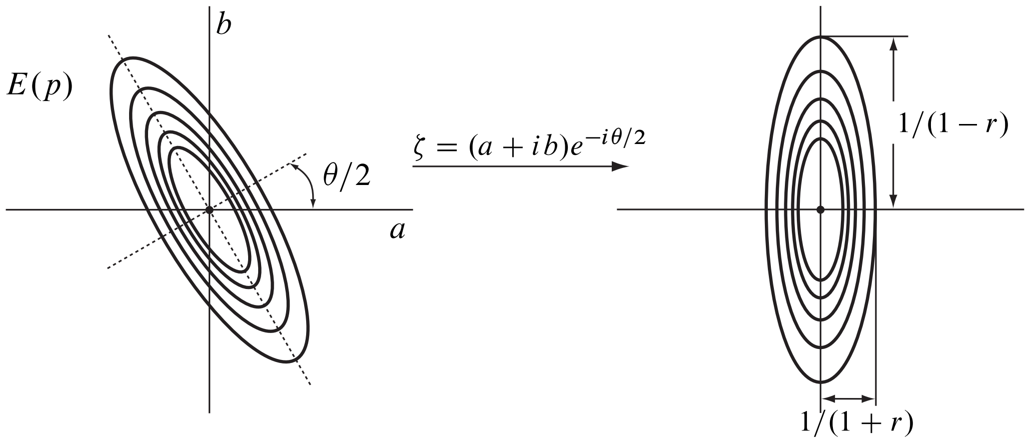

FIGURE 1.1. *Geometric interpretation of a Beltrami form as a field of ellipses on the tangent bundle.*

To summarize, we have at least three ways of thinking about Beltrami forms on a Riemann surface X: (i) as a $(-1, 1)$-tensor on X obeying the transformation rule (1.9); (ii) as a conformal structure on X; (iii) as a field of concentric ellipses on the real tangent bundle TX.

Now let X and Y be Riemann surfaces and $f : X \to Y$ be a diffeomorphism. Given a Beltrami form μ on Y, the ***pull-back*** $f^*\mu$ is defined as the Beltrami form of the pull-back metric $f^*\lambda$, where λ is any metric with the Beltrami form μ. It is easy to see that the definition is independent of the choice of λ. The pull-back operator on Beltrami forms has all the functorial properties of the similar operator on metrics. For example, if $f : X \to Y$ and $g : Y \to Z$ are given diffeomorphisms and μ is a Beltrami form on Z, then $(g \circ f)^*\mu = f^*(g^*\mu)$, or in short $(g \circ f)^* = f^* \circ g^*$.

Let us derive a local formula for the pull-back operator. Express $f : X \to Y$ locally as $w = f(z)$, where z and w are local coordinates on X and Y, and let

$\lambda = \gamma(w)\,|dw + \mu(w)\,d\bar{w}|$ be a metric on Y. Then

$$\begin{aligned} f^*\lambda &= \gamma(w(z))\,|w_z\,dz + w_{\bar z}\,d\bar z + \mu(w(z))\,(\overline{w}_z\,dz + \bar{w}_{\bar z}\,d\bar z)| \\ &= \gamma(w(z))\,|(w_z + \mu(w(z))\,\overline{w_{\bar z}})\,dz + (w_{\bar z} + \mu(w(z))\,\overline{w_z})\,d\bar z| \\ &= \gamma(w(z))\,(w_z + \mu(w(z))\,\overline{w_{\bar z}})\,\left|dz + \frac{w_{\bar z} + \mu(w(z))\,\overline{w_z}}{w_z + \mu(w(z))\,\overline{w_{\bar z}}}\,d\bar z\right|, \end{aligned}$$

where we have used the fact that

$$\bar{w}_z = \overline{w_{\bar z}} \qquad \text{and} \qquad \bar{w}_{\bar z} = \overline{w_z}.$$

It follows that

$$f^*\left(\mu(w)\,\frac{d\bar w}{dw}\right) = \frac{w_{\bar z} + \mu(w(z))\,\overline{w_z}}{w_z + \mu(w(z))\,\overline{w_{\bar z}}}\,\frac{d\bar z}{dz}. \tag{1.10}$$

In particular, the pull-back $\mu_f = f^*\mu_0$ of the zero Beltrami form on Y is given by

$$\mu_f = f^*(\mu_0) = \frac{w_{\bar z}}{w_z}\,\frac{d\bar z}{dz}. \tag{1.11}$$

The coefficient $\mu_f(z) = w_{\bar z}/w_z$ is called the ***complex dilatation*** of f in local coordinates z and w. By the Cauchy-Riemann equations, $\mu_f = \mu_0$ if and only if f is a conformal isomorphism.

Now (1.10) can be written as

$$f^*\left(\mu(w)\,\frac{d\bar w}{dw}\right) = \frac{\mu_f(z) + \mu(w(z))\,\frac{\overline{w_z}}{w_z}}{1 + \overline{\mu_f(z)}\mu(w(z))\,\frac{\overline{w_z}}{w_z}}\,\frac{d\bar z}{dz}. \tag{1.12}$$

Thus, at the level of coefficients, the pull-back operator on Beltrami forms acts pointwise as the disk automorphism

$$\mu \mapsto \frac{\tau(\mu\circ f) + \mu_f}{1 + \overline{\mu_f}\,\tau(\mu\circ f)}, \quad \text{where } \tau = \frac{\overline{w_z}}{w_z} \text{ has modulus 1.} \tag{1.13}$$

COROLLARY 1.3. *The pull-back operator $\mu \mapsto f^*\mu$ acts pointwise on the coefficients of Beltrami forms as an automorphism of the unit disk. When f is holomorphic, this automorphism reduces to a rotation about the origin. In particular, $f : X \to Y$ is a conformal isomorphism if and only if $f^*\mu_0 = \mu_0$.*

The central question now is

Question. Given a Beltrami form μ on a Riemann surface X, does there exist a new complex structure on X with respect to which μ is the zero Beltrami form?

If such a complex structure exists, we say that it is ***compatible*** with μ and we call μ ***integrable***. The surface X with the complex structure compatible with μ is denoted by X_μ. Here is an equivalent question

Question. Is there a Riemann surface Y and a diffeomorphism $f : X \to Y$ such that $f^*\mu_0 = \mu$?

Any such f is said to ***integrate*** μ. Geometrically, the condition means that the derivative Df pulls back the field of round circles on $T_{f(p)}Y$ (generated by μ_0) to the given field of ellipses on T_pX generated by μ. To see the equivalence of the two formulations, suppose $f : X \to Y$ integrates μ and let X_μ be X equipped with the pull-back of the complex structure of Y under f (whose local coordinates are the composition of f with the local coordinates of Y). Conversely, if X_μ exists, the identity map $X \to X_\mu$ clearly integrates μ.

If there are two diffeomorphisms $f : X \to Y$ and $g : X \to Z$ that integrate μ in the above sense, then $(g \circ f^{-1})^*\mu_0 = \mu_0$, which by Corollary 1.3 means $g \circ f^{-1} : Y \to Z$ is a conformal isomorphism. This shows that the Riemann surface Y in the integrability question is unique up to biholomorphism.

By (1.11) the integrability of Beltrami forms can be easily expressed in local coordinates. If $f : X \to Y$ integrates μ, then $\mu_f = f^*\mu_0 = \mu$. Writing f locally as $w = f(z)$, this translates into $\mu_f(z) = w_{\bar z}/w_z = \mu(z)$, or

$$w_{\bar z} = \mu\, w_z. \tag{1.14}$$

This partial differential equation is called the ***Beltrami equation***. When $\mu = \mu_0 = 0$, it reduces to the classical Cauchy-Riemann equation $w_{\bar z} = 0$. There is along history of attempts to solve the Beltrami equation under various regularity conditions on μ. The most basic result is the following

THEOREM 1.4 (Gauss)**.** *Suppose μ is a smooth complex-valued function defined in the unit disk $\mathbb{D}$ which satisfies $|\mu(z)| < 1$ at every $z \in \mathbb{D}$. Then there exists a diffeomorphism $f : \mathbb{D} \xrightarrow{\cong} f(\mathbb{D}) \subset \mathbb{C}$ such that $f_{\bar z} = \mu\, f_z$ in $\mathbb{D}$.*

Note that either $f(\mathbb{D}) = \mathbb{C}$ or $f(\mathbb{D})$ is conformally isomorphic to $\mathbb{D}$ by the Riemann Mapping Theorem. Thus, after post-composing with a biholomorphism if necessary, we can assume that f is a diffeomorphism $\mathbb{D} \to \mathbb{D}$ or $\mathbb{D} \to \mathbb{C}$; in other words, either $\mathbb{D}_\mu \cong \mathbb{C}$ or $\mathbb{D}_\mu \cong \mathbb{D}$.

EXAMPLE 1.5. Let $\mu(z) = k$ in $\mathbb{D}$ where $0 \le k < 1$ is a constant. The affine map

$$f : \mathbb{D} \to U := \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{(1+k)^2} + \frac{y^2}{(1-k)^2} < 1\right\}$$

defined by $w = f(z) = z + k\bar{z}$ integrates μ since clearly $w_{\bar{z}} = k\, w_z$. Post-composing w with a biholomorphism $U \to \mathbb{D}$ given by the Riemann Mapping Theorem, we obtain a diffeomorphism $\mathbb{D} \to \mathbb{D}$ which integrates μ. In particular, $\mathbb{D}_\mu \cong \mathbb{D}$.

EXAMPLE 1.6. Let $\mu(z) = z^2$ in $\mathbb{D}$. The diffeomorphism $f : \mathbb{D} \to \mathbb{C}$ defined by $w = f(z) = z/(1-|z|^2)$ integrates μ since

$$\frac{w_{\bar{z}}}{w_z} = \frac{\dfrac{z^2}{(1-|z|^2)^2}}{\dfrac{1}{(1-|z|^2)^2}} = z^2.$$

It follows that $\mathbb{D}_\mu \cong \mathbb{C}$.

The key difference between the above examples is that $\|\mu\|_\infty < 1$ in the former while $\|\mu\|_\infty = 1$ in the latter. It turns out that $(\mathbb{D}, \mu)$ is always isomorphic to $\mathbb{D}$ if $\|\mu\|_\infty < 1$. This will be a corollary of the measurable Riemann mapping theorem.

COROLLARY 1.7. *Every smooth Beltrami form on a Riemann surface is integrable.*

PROOF. Let μ be a smooth Beltrami form on a Riemann surface X. Cover X with countably many charts U_i with local coordinates $z_i : U_i \to \mathbb{D}$. Let μ_i be the pull-back of restriction $\mu|_{U_i}$ under z_i^{-1}. The Beltrami forms μ_i on $\mathbb{D}$ are integrable by Theorem 1.4, so there are diffeomorphisms $f_i : \mathbb{D} \to V_i := f_i(\mathbb{D}) \subset \mathbb{C}$ such that $f_i^{\,*}\mu_0 = \mu_i$. Now equip X with a new complex structure in which the $g_i := f_i \circ z_i : U_i \to V_i$ are the local coordinates. Note that the change of coordinates are indeed holomorphic since if $U_i \cap U_j \neq \emptyset$, then

$$\begin{aligned}(g_j \circ g_i^{-1})^*\mu_0 &= [(f_i^{-1})^* \circ (z_i^{-1})^* \circ z_j^* \circ f_j^*]\,\mu_0 \\ &= [(f_i^{-1})^* \circ (z_i^{-1})^* \circ z_j^*]\,\mu_j = [(f_i^{-1})^* \circ (z_i^{-1})^*]\,\mu \\ &= (f_i^{-1})^*\mu_i = \mu_0.\end{aligned}$$

Evidently, the Beltrami form μ in this new complex structure is identically zero since in U_i it is seen as $(g_i^{-1})^*\mu = \mu_0$. □

§1.2. Quasiconformal maps

In many applications, one is bound to consider conformal structures on Riemann surfaces which are only measurable. The integrability question for such conformal structures still makes sense, but maps that would integrate such structures can no longer be smooth. Simple examples show that measurable conformal structures are not generally integrable. However, with an extra boundedness assumption, they are integrable and the maps which integrate them are homeomorphisms which enjoy some degree of regularity. This leads to the idea of quasiconformal homeomorphisms between Riemann surfaces. In the category of quasiconformal maps rigidity and flexibility coexist, and that makes them an extremely powerful tool.

Roughly speaking, quasiconformal maps are a.e. differentiable homeomorphisms with bounded small-scale geometry. More specifically, a quasiconformal map $f : U \to V$ between planar domains has three essential features: (i) it is an orientation-preserving homeomorphism; (ii) it is differentiable at almost every $p \in U$ and the derivative $Df(p) : \mathbb{R}^2 \to \mathbb{R}^2$ is non-singular; (iii) $Df(p)$ pulls back round circles to ellipses whose eccentricity is bounded independent of p. This is in essence what quasiconformality means, except that the a.e. existence of the derivative should be replaced by a stronger condition called "ACL" (see Definition 1.8 below) or, equivalently, by f having locally integrable partial derivatives in the sense of distributions. Although a bit technical at first, this is just the right degree of regularity one needs to exclude pathologies and have a well-behaved class of maps.

It will be convenient to begin the discussion with quasiconformal maps between planar domains and consider the case of Riemann surfaces later.

DEFINITION 1.8. Let U and V be non-empty open sets in $\mathbb{C}$. An orientation-preserving homeomorphism $f : U \to V$ is called ***quasiconformal*** if

(i) f is ***absolutely continuous on lines (ACL)***. This means that for every closed rectangle $[a, b] \times [c, d] \subset U$, the restriction $x \mapsto f(x + iy)$ is absolutely continuous on $[a, b]$ for a.e. $y \in [c, d]$, and the restriction $y \mapsto f(x + iy)$ is absolutely continuous on $[c, d]$ for a.e. $x \in [a, b]$.

(ii) the partial derivatives of f (which by (i) exist a.e.) satisfy

$$|f_{\bar{z}}| \leq k|f_z| \qquad \text{a.e. in } U,$$

for some constant $0 \leq k < 1$.

To make the attribute more quantitative, we say that f is ***K-quasiconformal***, where $K := (1 + k)/(1 - k) \geq 1$.

EXAMPLE 1.9. Let $K \geq 1$ and define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(x + iy) := \begin{cases} x + iKy & \text{if } y \geq 0 \\ x + iy & \text{if } y < 0. \end{cases}.$$

Then f is ACL, with

$$f_z(x+iy) = \begin{cases} (1+K)/2 & \text{if } y > 0 \\ 1 & \text{if } y < 0 \end{cases} \quad \text{and} \quad f_{\bar{z}}(x+iy) = \begin{cases} (1-K)/2 & \text{if } y > 0 \\ 0 & \text{if } y < 0 \end{cases}$$

(note that the partial derivatives do not exist along the real axis unless $K = 1$). Thus, $|f_{\bar{z}}/f_z| \leq (K-1)/(K+1)$ and f is K-quasiconformal.

EXAMPLE 1.10. Let $0 \leq k < 1$ and define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) := \begin{cases} z + k\bar{z} & \text{if } |z| \leq 1 \\ z + k/z & \text{if } |z| > 1. \end{cases}$$

Then f is easily seen to be ACL, with

$$f_z(z) = \begin{cases} 1 & \text{if } |z| < 1 \\ 1 - k/z^2 & \text{if } |z| > 1 \end{cases} \quad \text{and} \quad f_{\bar{z}}(z) = \begin{cases} k & \text{if } |z| < 1 \\ 0 & \text{if } |z| > 1 \end{cases}$$

(again, the partial derivatives do not exist along the unit circle unless $k = 0$). It follows that $|f_{\bar{z}}/f_z| \leq k$, so f is K-quasiconformal with $K = (1+k)/(1-k)$.

Below we list some basic properties of quasiconformal maps. For a complete treatment and the proofs, we refer the reader to [**A**] or [**LV**]. Suppose $f : U \to V$ is K-quasiconformal. Then

(QC1) f is differentiable almost everywhere in U, that is, for a.e. $p \in U$,

$$f(p+z) = f(p) + f_z(p)\, z + f_{\bar{z}}(p)\, \bar{z} + \varepsilon(z),$$

where $\varepsilon(z)/z \to 0$ as $z \to 0$.

(QC2) The Jacobian

$$\operatorname{Jac}(f) = |f_z|^2 - |f_{\bar{z}}|^2$$

is positive a.e. and locally integrable in U, and the formula

$$\int_E \operatorname{Jac}(f)\, dx\, dy = \operatorname{area}(f(E))$$

holds for every measurable set $E \subset U$. In particular, f maps sets of area zero to sets of area zero.

(QC3) The partial derivatives f_z and $f_{\bar{z}}$ are locally square-integrable in U. In fact, since $|f_{\bar{z}}| \leq k|f_z|$ for $k = (K-1)/(K+1)$, the definition of $\operatorname{Jac}(f)$ shows that

$$|f_z|^2 \leq \frac{1}{1-k^2} \operatorname{Jac}(f) \qquad \text{and} \qquad |f_{\bar{z}}|^2 \leq \frac{k^2}{1-k^2} \operatorname{Jac}(f).$$

(QC4) The partial derivatives f_z and $f_{\bar{z}}$ are the distributional derivatives also, that is,

$$\int_U f_z\,\varphi = -\int_U f\,\varphi_z \qquad \text{and} \qquad \int_U f_{\bar{z}}\,\varphi = -\int_U f\,\varphi_{\bar{z}}$$

for every compactly supported smooth test function $\varphi : U \to \mathbb{C}$.

(QC5) For every topological annulus $A \subset U$,

$$K^{-1}\,\mathrm{mod}(A) \leq \mathrm{mod}(f(A)) \leq K\,\mathrm{mod}(A).$$

Surprisingly, this geometric condition is in fact equivalent to K-quasiconformality.

(QC6) The inverse $f^{-1} : V \to U$ is K-quasiconformal. This follows at once from (QC5).

(QC7) If $g : V \to W$ is K'-quasiconformal, the composition $g \circ f : U \to W$ is KK'-quasiconformal. This is also immediate from (QC5).

EXAMPLE 1.11. The unit disk $\mathbb{D}$ and the complex plane $\mathbb{C}$ are not quasiconformally homeomorphic: If there were a quasiconformal map $f : \mathbb{D} \to \mathbb{C}$, then by (QC5) the annulus $f(\{z : 1/2 < |z| < 1\})$ would have a finite modulus. But this annulus contains the punctured disk $\{z : |z| > r\}$ for a large r, whose modulus is infinite.

The following result will be frequently used:

THEOREM 1.12 (Weyl's Lemma)**.** *A* 1*-quasiconformal map is conformal.*

Note that 1-quasiconformality of f means that the Cauchy-Riemann equation $f_{\bar{z}} = 0$ holds a.e., and to conclude that f is conformal in this case, it is essential that we assume it is ACL.

EXAMPLE 1.13. Let $\xi : [0, 1] \to [0, 1]$ be any continuous non-decreasing function, with $\xi(0) = 0$, $\xi(1) = 1$, and $\xi'(x) = 0$ a.e. (the graph of such ξ is called a **devil's staircase**). Extend ξ to a map $\mathbb{R} \to \mathbb{R}$ by setting $\xi(x + n) = \xi(x) + n$ for $n \in \mathbb{Z}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(x + iy) := x + i(y + \xi(x)).$$

Then f is a homeomorphism which satisfies the Cauchy-Riemann equation $f_{\bar{z}} = 0$ a.e. in $\mathbb{C}$. However, f is not conformal. This does not contradict Weyl's Lemma since ξ fails to be absolutely continuous, so f is not ACL, hence not quasiconformal.

The standard chain rule formulas applies to compositions of quasiconformal maps. Suppose $f : U \to V$ and $g : V \to W$ are quasiconformal. By (QC7),

$g \circ f : U \to W$ is quasiconformal. By (QC1), there are sets of measure zero $A \subset U$ and $B \subset V$ away from which f and g are differentiable. Furthermore, by (QC2) and (QC6), $f^{-1}(B)$ has measure zero. Thus, $g \circ f$ is differentiable outside the measure zero set $A \cup f^{-1}(B)$ and the following chain rule formulas hold:

$$(g \circ f)_z = (g_z \circ f)\, f_z + (g_{\bar{z}} \circ f)\, \bar{f}_z$$
$$(g \circ f)_{\bar{z}} = (g_z \circ f)\, f_{\bar{z}} + (g_{\bar{z}} \circ f)\, \bar{f}_{\bar{z}}$$

If $f : U \to V$ is quasiconformal, (QC2) says that $\mathrm{Jac}(f) = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ a.e., so the partial derivative f_z must be non-zero a.e. in U. Inspired by the smooth case treated before, we call the measurable function

$$\mu_f := f_{\bar{z}}/f_z$$

the ***complex dilatation*** of f. The measurable function

$$K_f := \frac{1 + |\mu_f|}{1 - |\mu_f|} = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

is called the ***real dilatation*** of f. Thus f is K-quasiconformal precisely when $\|K_f\|_\infty \le K$ or $\|\mu_f\|_\infty \le k := (K-1)/(K+1)$.

The chain rule gives the following formula for the complex dilatation of the composition of quasiconformal maps:

$$\begin{aligned}
\mu_{g \circ f} &= \frac{(g_z \circ f)\, f_{\bar{z}} + (g_{\bar{z}} \circ f)\, \bar{f}_{\bar{z}}}{(g_z \circ f)\, f_z + (g_{\bar{z}} \circ f)\, \bar{f}_z} \\
&= \frac{f_{\bar{z}} + (\mu_g \circ f)\, \overline{f_z}}{f_z + (\mu_g \circ f)\, \overline{f_{\bar{z}}}} \\
&= \frac{\mu_f + (\mu_g \circ f)\, \overline{\frac{f_z}{f_z}}}{1 + \overline{\mu_f}\, (\mu_f \circ g)\, \overline{\frac{f_z}{f_z}}}
\end{aligned}$$

which, for good reason, is similar to the local formula for the pull-back operator in (1.13).

Here is a list of a few deeper properties of quasiconformal mappings:

(QC8) (Mori) If $f : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal with $f(0) = 0$, then

$$|f(z) - f(w)| \le 16\, |z - w|^{1/K} \qquad (z, w \in \mathbb{D}).$$

As a corollary, it follows that quasiconformal maps are locally Hölder: If $f : U \to V$ is K-quasiconformal, for every compact set $E \subset U$ there exists a constant $C = C(E, K) > 0$ such that

$$|f(z) - f(w)| \leq C\,|z - w|^{1/K} \qquad (z, w \in E).$$

(QC9) The family of all K-quasiconformal maps defined in a domain U which fix three marked points in U is compact. As a result, for any sequence $\{f_n\}$ of K-quasiconformal maps in U there is a sequence $\{\phi_n\}$ of Möbius maps such that $\{\phi_n \circ f_n\}$ converges locally uniformly in U to a K-quasiconformal map.

(QC10) (Astala) If $f : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal with $f(0) = 0$, then

$$\text{area}(f(E)) \leq C\ \text{area}(E)$$

for every measurable set $E \subset \mathbb{D}$. Here $C = C(K) > 0$. As a corollary, if $f : U \to V$ is K-quasiconformal, then $\text{Jac}(f) \in L^p_{\text{loc}}(U)$ for every $1 < p < K/(K-1)$.

(QC11) (Astala) Quasiconformal maps distort Hausdorff dimension by a bounded factor: If $f : U \to V$ is K-quasiconformal, $E \subset U$, $\dim(E) = \delta$ and $\dim(f(E)) = \delta'$, then

$$\frac{1}{K} \leq \frac{\frac{1}{\delta'} - \frac{1}{2}}{\frac{1}{\delta} - \frac{1}{2}} \leq K.$$

In particular, f preserves sets of dimension 0 and 2.

The transition to Riemann surfaces is now straightforward. An orientation-preserving homeomorphism $f : X \to Y$ between Riemann surfaces is K-quasiconformal if $w \circ f \circ z^{-1}$ is K-quasiconformal for each pair of local coordinates z on X and w on Y for which this composition makes sense. Much of the notions discussed above for diffeomorphisms, as well as the local computations, remain valid for quasiconformal maps, as they are differentiable almost everywhere. Thus, we can speak of measurable Riemannian metrics, conformal structures and Beltrami forms on Riemann surfaces, the pull-back of a such conformal structures or Beltrami forms under quasiconformal homeomorphisms, and the integrability condition for measurable Beltrami forms. It follows that if $f : X \to Y$ is a quasiconformal homeomorphism and μ is measurable Beltrami form on X, then

$$f \text{ integrates } \mu \iff \mu_f = f^*\mu_0 = \mu \iff f_{\bar{z}} = \mu\, f_z \text{ a.e.}$$

We say that a measurable conformal structure, or its associated Beltrami form μ has ***bounded dilatation*** if

$$\|\mu\|_\infty = \sup_{z \in X} |\mu(z)| < 1.$$

Thus, $f : X \to Y$ is quasiconformal if and only if the Beltrami form $\mu_f = f^*(\mu_0)$ has bounded dilatation.

The following theorem was proved by Morrey-Ahlfors-Bers:

THEOREM 1.14 (The measurable Riemann mapping theorem). *Let μ be a measurable Beltrami form on the Riemann sphere $\hat{\mathbb{C}}$ with $\|\mu\|_\infty = k < 1$. Then*

(i) There exists a unique quasiconformal homeomorphism $f = f^{[\mu]} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which fixes $0, 1, \infty$ and solves the Beltrami equation $\mu_f = \mu$.

(ii) f is a C^∞ (resp. real analytic) diffeomorphism if μ is C^∞ (resp. real analytic).

(iii) The normalized solution $f^{[t\mu]}$, given by part (i), depends holomorphically on the complex parameter $t \in \mathbb{D}(0, 1/k)$.

COROLLARY 1.15 (MRMT, the disk case). *Let μ be a measurable complex-valued function on the unit disk $\mathbb{D}$ with $\|\mu\|_\infty < 1$. Then there exists a quasiconformal homeomorphism $f : \mathbb{D} \to \mathbb{D}$ which satisfies the Beltrami equation $f_{\bar{z}} = \mu f_z$ almost everywhere. Moreover, f is unique up to post-composition with a conformal automorphism of $\mathbb{D}$.*

PROOF. Extend μ to $\hat{\mathbb{C}}$ by setting $\mu(z) = 0$ for $|z| > 1$ and apply Theorem 1.14. The restriction to $\mathbb{D}$ of the normalized solution given by that theorem will provide a quasiconformal solution $h : \mathbb{D} \to h(\mathbb{D})$ of the Beltrami equation. We have $h(\mathbb{D}) \neq \mathbb{C}$ since $\mathbb{C}$ and $\mathbb{D}$ are not quasiconformally homeomorphic, so by the Riemann mapping theorem there is a conformal map $g : h(\mathbb{D}) \to \mathbb{D}$. The composition $f = g \circ h$ will then be the required map. (Alternatively, extend μ to $\hat{\mathbb{C}}$ symmetrically by setting $\mu(z) = \mu(1/\bar{z})$ for $|z| > 1$, find the similar map h and use symmetry to show that $h(\mathbb{D}) = \mathbb{D}$ already.) □

The local case of the theorem easily yields the general version:

COROLLARY 1.16 (MRMT, the Riemann surface case). *Let μ be a measurable Beltrami form with bounded dilatation on a Riemann surface X. Then there exists a Riemann surface Y and a quasiconformal homeomorphism $f : X \to Y$ such that $\mu_f = f^*(\mu_0) = \mu$. If $g : X \to Z$ is another such homeomorphism, the map $g \circ f^{-1} : Y \to Z$ is a biholomorphism.*

Problems.

(1) Let $f : U \to V$ be quasiconformal. Show that

$$\mu_{f^{-1}} \circ f = -\frac{\overline{f_z}}{f_z}\, \mu_f .$$

(2) Suppose μ is a Beltrami form on a Riemann surface X that is ***locally of bounded dilatation*** in the sense that

$$\sup_{z \in K} |\mu|(z) < 1$$

for every compact set $K \subset X$. Show that μ is integrable in the following sense: There is a Riemann surface Y and a homeomorphism $f : X \to Y$ such that (i) each $p \in X$ has a neighborhood U_p for which $f|_{U_p}$ is quasiconformal; (ii) $f^*\mu_0 = \mu$.

(3) Let μ be a smooth Beltrami form in $\mathbb{D}$ which is rotationally symmetric so that $|\mu|$ depends only on $r = |z|$. Show that the Riemann surface $\mathbb{D}_\mu$ is conformally isomorphic to $\mathbb{D}$ or $\mathbb{C}$ according as

$$\lim_{a \to 1^-} \int_{\frac{1}{2}}^{a} \frac{1 + |\mu|}{1 - |\mu|}\, \frac{dr}{r}$$

is finite or infinite.

(4) Study compactness properties of $\mathrm{QS}_0(M)$.

(5) Every homeomorphism $h : \mathbb{R} \to \mathbb{R}$ can be written as $h(x) = h(0) + \int_0^x d\mu$ for some positive measure μ without atoms. Show that h is quasisymmetric iff μ is a doubling measure.

Research problem.

(1) Let μ be a smooth Beltrami form in $\mathbb{D}$. Develop a method for deciding whether the Riemann surface $\mathbb{D}_\mu$ is conformally isomorphic to $\mathbb{D}$ or $\mathbb{C}$.

CHAPTER 2

Some Applications of Quasiconformal Maps

§2.1. The diffeomorphism group of the 2-sphere

We begin our application tour by proving a classical result on the homotopy type of the diffeomorphism group of the 2-sphere [**S**]:

THEOREM 2.1 (Smale)**.** *The group* $\mathrm{Diff}^+(\mathbb{S}^2)$ *of smooth orientation-preserving diffeomorphisms of the* 2*-sphere has a strong deformation retraction onto the rotation subgroup* SO(3).

Recall that a space X has a ***strong deformation retraction*** onto a subspace Y if there is a continuous map $F : X \times [0, 1] \to X$ such that

- $F(x, 0) = x$ and $F(x, 1) \in Y$ for all $x \in X$;
- $F(y, t) = y$ for all $y \in Y$ and $t \in [0, 1]$.

It is easily seen that in this case X and Y have the same homotopy type; in fact, the inclusion $Y \hookrightarrow X$ is a homotopy equivalence. Thus, Smale's theorem shows that $\mathrm{Diff}^+(\mathbb{S}^2)$ has the homotopy type of a finite cell-complex. This is also true for $\mathrm{Diff}^+(\mathbb{S}^3)$ (Hatcher) and is known to be false for all dimensions ≥ 7.

Let us identify $\mathbb{S}^2$ with the Riemann sphere $\hat{\mathbb{C}}$ via, say, the stereographic projection. The idea of the proof consists of first constructing a strong deformation retraction from $\mathrm{Diff}^+(\hat{\mathbb{C}})$ onto the Möbius group $\mathrm{Aut}(\hat{\mathbb{C}})$ and then follow it by another from $\mathrm{Aut}(\hat{\mathbb{C}})$ onto SO(3).

PROOF. Let $\varphi \in \mathrm{Diff}^+(\hat{\mathbb{C}})$ and $\mu = \mu_\varphi$, so $\|\mu\|_\infty < 1$. Let $\Phi \in \mathrm{Aut}(\hat{\mathbb{C}})$ be the unique Möbius map which agrees with φ on the set $\{0, 1, \infty\}$. Consider the smooth Beltrami forms $t\mu$ on $\hat{\mathbb{C}}$ for $0 \leq t \leq 1$, and set $\varphi_t := \Phi \circ \varphi^{[t\mu]}$ (here $\varphi^{[t\mu]} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the unique solution of the Beltrami equation $w_{\bar{z}} = (t\mu)\, w_z$ which fixes $0, 1, \infty$). Then $t \mapsto \varphi_t$ is a continuous path in $\mathrm{Diff}^+(\hat{\mathbb{C}})$ that connects $\varphi_0 = \Phi$ to $\varphi_1 = \varphi$. The map $F : \mathrm{Diff}^+(\hat{\mathbb{C}}) \times [0, 1] \to \mathrm{Diff}^+(\hat{\mathbb{C}})$ defined by $F(\varphi, t) := \varphi_{1-t}$ is a strong deformation retraction onto $\mathrm{Aut}(\hat{\mathbb{C}})$.

Next we construct a strong deformation retraction $G : \mathrm{Aut}(\hat{\mathbb{C}}) \times [0, 1] \to \mathrm{Aut}(\hat{\mathbb{C}})$ onto the rotation subgroup SO(3). The map $H : \mathrm{Diff}^+(\hat{\mathbb{C}}) \times [0, 1] \to \mathrm{Diff}^+(\hat{\mathbb{C}})$ defined by

$$H(\varphi, t) := \begin{cases} F(\varphi, 2t) & t \in [0, 1/2] \\ G(F(\varphi, 1), 2t - 1) & t \in [1/2, 1] \end{cases}$$

will then be a strong deformation retraction of $\mathrm{Diff}^+(\hat{\mathbb{C}})$ onto SO(3).

LEMMA 2.2. *Every Möbius map* $\varphi \in \mathrm{Aut}(\hat{\mathbb{C}})$ *can be decomposed uniquely as*

$$\varphi = \rho \circ \alpha \circ \beta,$$

where $\rho \in \mathrm{SO}(3)$, $\alpha(z) = rz$ *for some* $r > 0$, *and* $\beta(z) = z + \tau$ *for some* $\tau \in \mathbb{C}$.

Assuming this lemma for a moment, it is easy to construct G: Given $\varphi \in \mathrm{Aut}(\hat{\mathbb{C}})$, take the decomposition $\varphi = \rho \circ \alpha \circ \beta$ as in the lemma, and define

$$G(\varphi, t) := \rho \circ \alpha_t \circ \beta_t,$$

where $\alpha_t(z) := r^{1-t} z$ and $\beta_t(z) := z + (1 - t)\tau$. This proves Theorem 2.1. □

PROOF OF LEMMA 2.2: The decomposition is immediate when $\varphi(\infty) = \infty$, for in this case $\varphi(z) = az + b$ for some $a, b \in \mathbb{C}$ with $a \neq 0$, and we can take

$$\rho(z) = \frac{a}{|a|} z, \quad \alpha(z) = |a|z, \quad \text{and} \quad \beta(z) = z + \frac{b}{a}.$$

For the general case, first post-compose φ with an element of SO(3) to keep ∞ fixed, and then decompose the resulting map as above.

To show uniqueness, suppose φ has two such decompositions, say

$$\varphi = \rho_1 \circ \alpha_1 \circ \beta_1 = \rho_2 \circ \alpha_2 \circ \beta_2,$$

where $\alpha_i(z) = r_i z$ and $\beta_i = z + \tau_i$. The map

$$\rho_2^{-1} \circ \rho_1 = \alpha_2 \circ \beta_2 \circ (\alpha_1 \circ \beta_1)^{-1} : z \mapsto \frac{r_2}{r_1} z + r_2(\tau_2 - \tau_1)$$

is in SO(3) and affine, so it can only be a rotation around the origin. This implies

$$\left| \frac{r_2}{r_1} \right| = 1 \quad \text{and} \quad r_2(\tau_2 - \tau_1) = 0,$$

which gives $r_1 = r_2$ and $\tau_1 = \tau_2$. □

§2.2. Quasiconformal conjugacy classes of rational maps

Two rational maps f, g of the Riemann sphere are ***quasiconformally conjugate*** if there is a quasiconformal map $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which satisfies

$$\varphi \circ f = g \circ \varphi.$$

This is clearly an equivalence relation, and each equivalence class under this relation is called a ***quasiconformal conjugacy class***.

Suppose f, g are quasiconformally conjugate by φ as above. Then the Beltrami form $\mu = \mu_\varphi$ is f-invariant. This is simply because g is holomorphic:

$$\begin{aligned} f^*\mu = f^*(\varphi^*\mu_0) &= (\varphi \circ f)^*\mu_0 = (g \circ \varphi)^*\mu_0 \\ &= \varphi^*(g^*\mu_0) = \varphi^*\mu_0 = \mu. \end{aligned}$$

Conversely, suppose μ is an f-invariant Beltrami differential with $\|\mu\|_\infty < 1$. Then a similar argument shows that the branched covering $g = \varphi^{[\mu]} \circ f \circ (\varphi^{[\mu]})^{-1}$ preserves μ_0, so it is locally 1-quasiconformal away from the branch points, hence is a rational map.

Thus, there is a correspondence between f-invariant Beltrami forms of bounded dilatation and rational maps which are quasiconformally conjugate to f (the correspondence need not be one-to-one).

As a basic dynamical application of the preceding remark, we show

THEOREM 2.3. *The quasiconformal conjugacy class of a rational map f is always path-connected.*

Of course this conjugacy class could reduce to a point, in which case f is called ***quasiconformally rigid***. For example, in the family of normalized quadratic polynomials $\{f_c(z) = z^2 + c\}_{c\in\mathbb{C}}$, the map f_c is quasiconformally rigid if and only if c belongs to the boundary of the Mandelbrot set or is the center of the hyperbolic component.

PROOF. Suppose φ is a quasiconformal conjugacy between f and another rational map g and set $\mu = \mu_\varphi$. Then, for every $0 \leq t \leq 1$,

$$f^*(t\mu) = tf^*\mu = t\mu.$$

Here we have used the fact that the pull-back operator f^* acts as a rotation about the origin and hence is linear. As before, set $\varphi_t = \Phi \circ \varphi^{[t\mu]}$ where Φ is the unique Möbius map which agrees with φ on $\{0, 1, \infty\}$. Then $t \mapsto g_t := \varphi_t \circ f \circ (\varphi_t)^{-1}$ is a path within the quasiconformal conjugacy class of f which connects $g_0 =$

$\Phi \circ f \circ \Phi^{-1}$ to $g_1 = g$. Joining this path to $t \mapsto \Phi_t \circ f \circ \Phi_t^{-1}$ in which $t \mapsto \Phi_t$ is a path in $\mathrm{Aut}(\hat{\mathbb{C}})$ which connects id to Φ, we obtain a path from f to g. $\square$

§2.3. Local fixed-point theory

Let $f(z) = \lambda z + O(z^2)$ be the germ of a holomorphic map in the plane fixing the origin. The multiplier $\lambda = f'(0)$ is clearly invariant under smooth conjugacies. On the other hand, $z \mapsto 2z$ is topologically (even quasiconformally) conjugate to $z \mapsto 3z$.

A remarkable theorem of Naishul asserts that when the origin is an indifferent fixed point in the sense that $|\lambda| = 1$, then the multiplier λ is invariant under *topological* conjugacies. Here we prove a weaker version of this result by using the measurable Riemann mapping theorem.

THEOREM 2.4. *Let $f(z) = \lambda z + O(z^2)$ and $g(z) = \nu z + O(z^2)$ be quasiconformally conjugate near* 0*. If $|\lambda| = 1$, then $\lambda = \nu$.*

PROOF. Let φ be a quasiconformal homeomorphism defined near 0 which satisfies $\varphi(0) = 0$ and $\varphi \circ f = g \circ \varphi$. Consider the Beltrami form $\mu = \mu_\varphi$ defined near the origin, which is clearly f-invariant. Let $\delta > 0$ be small and extend μ to $\mathbb{C}$ by setting $\mu(z) = 0$ for $|z| > \delta$. Consider the Beltrami forms $t\mu$ for $|t| < 1/\|\mu\|_\infty$. Since $f^*\mu = \mu$ and f^* is linear, it follows that $f^*(t\mu) = t\mu$ near 0. Let $\varphi_t = \varphi^{[t\mu]}$. Then $g_t = \varphi_t \circ f \circ \varphi_t^{-1}$ is a 1-quasiconformal homeomorphism near the origin, hence holomorphic there. Moreover, $t \mapsto g_t(z)$ is holomorphic for each fixed z sufficiently close to 0 [Caution: This is true but non-trivial, as the inverse φ_t^{-1} may not depend holomorphically on t]. Writing $g_t(z) = \lambda_t z + O(z^2)$, it follows that $t \mapsto \lambda_t$ is also holomorphic. But g_t is conjugate to f whose fixed point at $z = 0$ is indifferent, so $|\lambda_t| = 1$ for all $t \in \mathbb{D}(0, 1 + \varepsilon)$. The open mapping theorem now implies that $t \mapsto \lambda_t$ is constant. Since $\varphi_0 = \mathrm{id}$, we have $g_0 = f$ so $\lambda_0 = \lambda$. Similarly, $\varphi_1 \circ \varphi^{-1}$ is conformal, so g_1 is holomorphically conjugate to g, so $\lambda_1 = \nu$. We conclude that $\lambda = \nu$. $\square$

As another application, consider the problem of linearizing holomorphic maps near attracting or repelling fixed points. A classical theorem of Koenigs asserts that every holomorphic germ $f(z) = \lambda z + O(z^2)$ with $|\lambda| \neq 0, 1$ is holomorphically linearizable. The classical proof, for $|\lambda| < 1$, consists of showing that the sequence $\{\lambda^{-n} f^{\circ n}(z)\}_{n \geq 1}$ converges uniformly in a neighborhood of the origin to a holomorphic map Φ. It is then clear that $\Phi'(0) = 1$ and $\Phi(f(z)) = \lambda\Phi(z)$. Here we give a proof of this result by applying the measurable Riemann mapping theorem.

THEOREM 2.5 (Koenigs). *If $f(z) = \lambda z + O(z^2)$ is a holomorphic germ with $|\lambda| \neq 0, 1$, there exists a holomorphic change of coordinate $z \mapsto \Phi(z)$ defined near the origin such that $\Phi(0) = 0$ and $\Phi(f(z)) = \lambda\Phi(z)$.*

PROOF. Without losing generality assume $|\lambda| < 1$ (otherwise consider the local inverse of f). Choose a disk U centered at 0, small enough so that $f(U)$ is compactly contained in U. It follows by induction that $f^{\circ n}(U)$ is compactly contained in $f^{\circ n-1}(U)$ for all $n \geq 1$, and that $\operatorname{diam} f^{\circ n}(U) \to 0$ as $n \to \infty$. Let L denote the linear contraction $z \mapsto z/2$. Take a smooth diffeomorphism $\psi : A = \{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\} \to \overline{U \smallsetminus f(U)}$ subject only to the condition $\psi(L(z)) = f(\psi(z))$ for $|z| = 1$. Extend ψ to a homeomorphism $\overline{\mathbb{D}} \to \overline{U}$ by defining $\psi(0) = 0$ and $\psi(L^{\circ n}(z)) = f^{\circ n}(\psi(z))$ for all $n \geq 1$ and all $z \in A$. Then ψ is quasiconformal and satisfies

$$\psi(L(z)) = f(\psi(z)) \qquad \text{for all } z \in \mathbb{D}.$$

Now consider the Beltrami form $\mu = \mu_\psi$ on $\mathbb{D}$. Extend μ to the entire plane by taking pull-backs under L. The resulting Beltrami form (still denoted by μ) is easily seen to be L-invariant and with bounded dilatation. The quasiconformal map $g = \varphi^{[\mu]} \circ L \circ (\varphi^{[\mu]})^{-1} : \mathbb{C} \to \mathbb{C}$ preserves μ_0, hence is holomorphic. Since $g(0) = 0$, we must have $g(z) = \nu z$ for some $\nu \neq 0$.

Set $\Phi = \varphi \circ \psi^{-1}$. Then Φ is a 1-quasiconformal homeomorphism defined in a neighborhood of the fixed point 0. By Weyl's Lemma, Φ is holomorphic. Moreover, Φ conjugates f to g near 0, so $\nu = g'(0) = f'(0) = \lambda$. □

§2.4. Quasiconformal surgery on rational maps

Here is an outline of the construction of Herman rings by surgery, following Shishikura. Suppose f is a rational map of degree $d \geq 2$ with a fixed Siegel disk Δ of rotation number θ. Take another rational map g of degree $d' \geq 2$ with a fixed Siegel disk Δ' of rotation number $-\theta$. We will construct a rational map F, of degree $d + d' - 1$, with a Herman ring of rotation number θ. The idea is to cut out invariant disks from Δ and Δ' and paste the resulting spheres-with-hole along their boundaries to obtain a sphere. There is an obvious action on this sphere coming from the action of f and g on the pieces. We apply the measurable Riemann mapping theorem to realize this action as a rational map.

More precisely, let $\phi : \Delta \xrightarrow{\cong} \mathbb{D}(0,2)$ and $\psi : \Delta' \xrightarrow{\cong} \mathbb{D}(0,2)$ be conformal isomorphisms which satisfy

$$\phi(f(z)) = e^{2\pi i\theta}\phi(z) \quad \text{and} \quad \psi(g(z)) = e^{-2\pi i\theta}\psi(z).$$

Consider the invariant Jordan curves

$$\gamma = \{z \in \Delta : |\phi(z)| = 1\} \quad \text{and} \quad \gamma' = \{z \in \Delta' : |\psi(z)| = 1\}.$$

The map $h : \gamma \to \gamma'$ defined by $h(z) = \psi^{-1}(\overline{\phi(z)})$ is a smooth orientation-reversing diffeomorphism which satisfies $h(f(z)) = g(h(z))$ for all $z \in \gamma$. Extend h to a quasiconformal homeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the following properties:

- h maps $\mathrm{int}(\gamma)$ to $\mathrm{ext}(\gamma')$ and $\mathrm{ext}(\gamma)$ to $\mathrm{int}(\gamma')$. (Here “int” refers to the complementary component of the invariant curve which contains the center of the Siegel disk and “ext” refers to the other component.)
- h is conformal in a neighborhood of $\hat{\mathbb{C}} \smallsetminus (\Delta \cap h^{-1}(\Delta'))$.

Define

$$\tilde{F}(z) = \begin{cases} f(z) & \text{if } z \in \gamma \cup \mathrm{ext}(\gamma) \\ (h^{-1} \circ g \circ h)(z) & \text{if } z \in \mathrm{int}(\gamma) \end{cases}$$

It is easy to check that $\tilde{F}$ is a degree $d + d' - 1$ branched covering of the sphere which is locally quasiconformal away from its branch points. Moreover, $A = \Delta \cap h^{-1}(\Delta')$ is a “topological Herman ring” of rotation number θ for $\tilde{F}$, and F is holomorphic in a neighborhood of $\hat{\mathbb{C}} \smallsetminus F^{-1}(A)$.

To conjugate $\tilde{F}$ to a rational map, define a Beltrami form μ on $\hat{\mathbb{C}}$ as follows. First define μ on A by

$$\mu = \begin{cases} \mu_0 & \text{on } A \cap \mathrm{ext}(\gamma) \\ h^*\mu_0 & \text{on } A \cap \mathrm{int}(\gamma) \end{cases}$$

Clearly, $\tilde{F} : A \to A$ preserves μ. Extend μ to the union $\bigcup_{n \geq 1} \tilde{F}^{-n}(A)$ by pulling back via the appropriate iterate of $\tilde{F}$. Note that only the first pull-back to $\tilde{F}^{-1}(A) \smallsetminus A$ can possibly increase the dilatation of μ; all further pull-backs are taken by iterates of $\tilde{F}$ which are holomorphic and so do not change the dilatation. On the complement of this union, set $\mu = \mu_0$. The Beltrami form μ defined this way is clearly $\tilde{F}$-invariant and has bounded dilatation. It follows that $F = \varphi^{[\mu]} \circ \tilde{F} \circ (\varphi^{[\mu]})^{-1}$ is a rational map with a Herman ring $\varphi^{[\mu]}(A)$ of rotation number θ.

§2.5. The no wandering domain theorem

We present a simplified version of Sullivan’s proof of Fatou’s no wandering domain conjecture, following N. Baker and C. McMullen.

THEOREM 2.6 (Sullivan)**.** *Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. Then every Fatou component U of f is eventually periodic, i.e., there exist $n > m > 0$ such that $f^{\circ n}(U) = f^{\circ m}(U)$.*

The idea of the proof is as follows: Assuming there exists a ***wandering*** Fatou component U (or simply a ***wandering domain***), we change the conformal structure of the sphere along the grand orbit of U to find an infinite-dimensional family of rational maps of degree d, all quasiconformally conjugate to f. This is a contradiction since the space Rat_d of rational maps of degree d, as a Zariski open subset of $\mathbb{CP}^{2d+1}$, is finite-dimensional. The eventual periodicity statement for entire maps is false. For example, the transcendental map $z \mapsto z + \sin(2\pi z)$ has wandering domains.

LEMMA 2.7 (Baker)**.** *If U is a wandering domain for a rational map f, then $f^{\circ n}(U)$ is simply connected for all large n.*

PROOF. Let $U_n = f^{\circ n}(U)$. Replacing U by U_k for some large k if necessary, we may assume that no U_n contains a critical point of f, so that $f^{\circ n} : U \to U_n$ is a covering map for all n. We can also arrange that $\infty \in U$. Then the U_n are disjoint subsets of $\mathbb{C} \smallsetminus U$ for $n \geq 1$, so area$(U_n) \to 0$ as $n \to \infty$. But $\{f^{\circ n}|_U\}$ is a normal family, so every convergent subsequence of this sequence must be a constant function. In particular, diam$(f^{\circ n}(K)) \to 0$ for every compact set $K \subset U$. Take any loop $\gamma \subset U$ and set $\gamma_n = f^{\circ n}(\gamma) \subset U_n$. By the above argument diam$(\gamma_n) \to 0$. If B_n is the union of the bounded components of $\mathbb{C} \smallsetminus \gamma_n$, it follows that diam$(B_n) \to 0$ also. Since $f(B_n)$ is open, $\partial f(B_n) \subset \gamma_{n+1}$, and diam $f(B_n) \to 0$, we must have $f(B_n) \subset \overline{B_{n+1}}$ for large n. In particular, the iterated images of B_n are subsets of $\mathbb{C} \smallsetminus U$ for large n. Montel's theorem then implies $B_n \subset F(f)$, which gives $B_n \subset U_n$. Thus γ_n is null-homotopic in U_n for large n. Since $f^{\circ n} : U \to U_n$ is a covering map, we can lift this homotopy to U, which proves U is simply connected. $\square$

Let a rational map f have a wandering domain U. In view of the above lemma, we can assume that $U_n = f^{\circ n}(U)$ is simply connected and $f : U_n \to U_{n+1}$ is a conformal isomorphism for all $n \geq 0$. Given an L^∞ Beltrami form μ defined in U, we can construct an f-invariant L^∞ Beltrami form on $\hat{\mathbb{C}}$ as follows. Use the forward and backward iterates of f to spread μ along the grand orbit

$$\mathrm{GO}(U) = \{z \in \hat{\mathbb{C}} : f^{\circ n}(z) \in U_m \text{ for some } n, m \geq 0\}.$$

On the complement $\hat{\mathbb{C}} \smallsetminus \mathrm{GO}(U)$, set $\mu = \mu_0$. The resulting Beltrami form is defined a.e. on $\widehat{\mathbb{C}}$, it satisfies $f^*\mu = \mu$ by the way it is defined, and $\|\mu\|_\infty < \infty$

since spreading $\mu|_U$ along $\mathrm{GO}(U)$ by the iterates of the holomorphic map f does not change the dilatation. Now consider the deformation $t\mu$ for complex t with $|t| < \varepsilon$, where $\varepsilon > 0$ is small enough to guarantee $\|t\mu\|_\infty < 1$. Since f is holomorphic, f^* acts as a linear rotation, so $f^*(t\mu) = t\mu$. Let $\varphi_t = \varphi^{[t\mu]} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the unique solution of the Beltrami equation $w_{\bar{z}} = (t\mu)\, w_z$ which fixes $0, 1, \infty$. Then $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$ is a rational map of degree d and $t \mapsto f_t$ is holomorphic, with $f_0 = f$. The infinitesimal variation

$$w(z) = \frac{d}{dt}\Big|_{t=0} f_t(z)$$

defines a holomorphic vector field whose value at z lies in the tangent space $T_{f(z)}\widehat{\mathbb{C}}$. In other words, w is a holomorphic section of the pull-back bundle $f^*(T\widehat{\mathbb{C}})$ which in turn can be identified with a tangent vector in $T_f\,\mathrm{Rat}_d$. This is the so-called ***infinitesimal deformation*** of f induced by μ. We say that μ induces a ***trivial*** deformation if $w = 0$ everywhere.

Another way of describing w is as follows: First consider the unique quasiconformal vector field solution to the equation

$$\overline{\partial} v = \mu \qquad \text{with } v(0) = v(1) = v(\infty) = 0.$$

This is precisely the infinitesimal variation $(d/dt)|_{t=0}\, \varphi_t(z)$ of the normalized solution of the Beltrami equation. It is not hard to check that $w = \delta_f v$, where

$$\delta_f v(z) = v(f(z)) - f'(z)v(z)$$

measures the deviation of v from being f-invariant. Note in particular that $\delta_f v$ is holomorphic even though v is only quasiconformal, and that $w = \delta_f v$ depends linearly on μ, a fact that is not immediately clear from the first description of w. It follows that μ induces a trivial deformation if and only if v is f-invariant.

The triviality condition $\delta_f v = 0$ forces v to vanish on the Julia set $J(f)$. In fact, let $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$ be a repelling cycle of f with multiplier λ. Then the condition $\delta_f v = 0$ implies $v(z_{j+1}) = f'(z_j)v(z_j)$ for all $j = 0, \ldots, n-1$, so

$$\prod_{j=0}^{n-1} v(z_j) = \lambda \cdot \prod_{j=0}^{n-1} v(z_j).$$

The assumption $|\lambda| > 1$ then implies $v(z_j) = 0$ for some j, hence for all j. Since $J(f)$ is the closure of repelling cycles and v is continuous, we conclude that $v(z) = 0$ for all $z \in J(f)$.

The above construction gives well-defined linear maps

$$B(U) \overset{i}{\hookrightarrow} B(\widehat{\mathbb{C}}, f) \xrightarrow{D} T_f \operatorname{Rat}_d$$

Here $B(U)$ is the space of L^∞ Beltrami forms in U, $B(\widehat{\mathbb{C}}, f)$ is the space of f-invariant L^∞ Beltrami forms on $\widehat{\mathbb{C}}$, and D is the linear operator $D\mu = w = \delta_f v$ constructed above.

LEMMA 2.8. *$B(U)$ contains an infinite-dimensional subspace $N(U)$ of compactly supported Beltrami forms with the following property: If $\mu \in N(U)$ satisfies $\mu = \overline{\partial} v$ for some quasiconformal vector field v with $v|_{\partial U} = 0$, then $\mu = 0$.*

Assuming this lemma for a moment, let us see how we can prove Theorem 2.6. Consider the subspace $N(U)$ for the simply connected wandering domain U and restrict the above diagram to this subspace. If $D(\mu) = 0$ for some $\mu \in N(U)$, or in other words if μ induces a trivial deformation, that means the normalized solution v to $\overline{\partial} v = \mu$ is f-invariant. Hence $v = 0$ on $J(f)$ and in particular on the boundary of U. By the property of $N(U)$, $\mu = 0$. This means that the infinite-dimensional subspace $N(U)$ injects into $T_f \operatorname{Rat}_d$ whose dimension is $2d + 1$. The contradiction shows that no wandering domain can exist.

It remains to prove Lemma 2.8. Let us first consider the corresponding problem for the unit disk $\mathbb{D}$. Let $\tilde{N}(\mathbb{D}) \subset B(\mathbb{D})$ be the linear span of the Beltrami forms $\mu_k(z) = \bar{z}^k \frac{d\bar{z}}{dz}$ for $k \geq 0$. The vector field

$$V_k(z) = \begin{cases} \frac{1}{k+1}\, \bar{z}^{k+1} \frac{\partial}{\partial z} & |z| < 1 \\ \frac{1}{k+1}\, z^{-(k+1)} \frac{\partial}{\partial z} & |z| \geq 1 \end{cases}$$

solves the equation $\overline{\partial} V_k = \mu_k$ in $\mathbb{D}$. Let $\mu = \overline{\partial} v \in \tilde{N}(\mathbb{D})$ and $v|_{\partial\mathbb{D}} = 0$, and take the appropriate linear combination V of the V_k which solves $\overline{\partial} V = \mu$. Then $V - v$ is holomorphic in $\mathbb{D}$ and agrees with V on the boundary $\partial\mathbb{D}$. This is impossible if $V|_{\partial\mathbb{D}}$ has any negative power of z in it. Hence $\mu = 0$. To get the compact support condition, let $N(\mathbb{D}) \subset B(\mathbb{D})$ be obtained by restricting elements of $\tilde{N}(\mathbb{D})$ to the disk $|z| < 1/2$ and extending them to be zero on $1/2 \leq |z| < 1$. If $\mu = \overline{\partial} v \in N(\mathbb{D})$ and $v|_{\partial\mathbb{D}} = 0$, then v has to vanish in the annulus $1/2 < |z| < 1$ since it is holomorphic there. In particular, it is zero on $|z| = 1/2$. Now the same argument applied to the disk $|z| < 1/2$ shows $\mu = 0$.

For the general case, consider a conformal isomorphism $\psi : \mathbb{D} \xrightarrow{\cong} U$ with the inverse $\phi = \psi^{-1}$ and define $N(U) = \phi^*(N(\mathbb{D}))$. Let v be a quasiconformal vector field such that $\mu = \overline{\partial} v \in N(U)$ and $v|_{\partial U} = 0$. Then $\phi_*(v) = (v \circ \psi)/\psi'$ is a quasiconformal vector field in $\mathbb{D}$ which is holomorphic near the boundary $\partial\mathbb{D}$ and $v(\psi(z)) \to 0$ as $|z| \to 1$. By the reflection principle, $v \circ \psi$ is identically zero near $\partial\mathbb{D}$. Since $\psi^*\mu = \overline{\partial}\, \phi_*(v) \in N(\mathbb{D})$, we must have $\psi^*\mu = 0$, which implies $\mu = 0$.

REMARK 2.9. Sullivan's original argument had to deal with two essential difficulties: (i) the possibility of U being non simply connected, perhaps of infinite topological type; (ii) the possible complications near the boundary of U, for example when ∂U is not locally-connected. He addressed the former by using a direct limit argument, and the latter by using Carathéodory's theory of "prime ends." Both of these difficulties are miraculously bypassed in the present proof.

§2.6. Holomorphic motions

DEFINITION 2.10. Let $A \subset \hat{\mathbb{C}}$ be a set with at least 4 points and T be a connected complex manifold with base point t_0. A ***holomorphic motion of*** A ***over*** (T, t_0) is a map $\varphi : T \times A \to \hat{\mathbb{C}}$ such that

(i) $z \mapsto \varphi(t, z)$ is injective for each $t \in T$.

(ii) $t \mapsto \varphi(t, z)$ is holomorphic for each $z \in A$.

(iii) $\varphi(t_0, z) = z$ for every $z \in A$.

In other words, $\{\varphi_t(\cdot) = \varphi(t, \cdot)\}_{t \in T}$ is a holomorphic family of injections of A into $\hat{\mathbb{C}}$, with $\varphi_{t_0} = \mathrm{id}_A$.

A few remarks on this definition are in order:

- There is no assumption on the joint continuity of φ in (t, z), or even continuity in z for fixed t. They follow automatically from the λ-Lemma to be discussed below.
- For our purposes, we usually take $(T, t_0) = (\mathbb{D}, 0)$ and call φ a holomorphic motion over $\mathbb{D}$.
- We can always assume that the motion is *normalized* in the sense that $0, 1, \infty$ belong to A and they remain fixed under the motion. To see this, take distinct points z_1, z_2, z_3 in A and let $\alpha, \beta_t \in \mathrm{Aut}(\hat{\mathbb{C}})$ be determined by

$$\alpha(0) = z_1 \quad \alpha(1) = z_2 \quad \alpha(\infty) = z_3$$

and

$$\beta_t(\varphi_t(z_1)) = 0 \quad \beta_t(\varphi_t(z_2)) = 1 \quad \beta_t(\varphi_t(z_3)) = \infty.$$

Then $\psi_t = \beta_t \circ \varphi_t \circ \alpha$ is a normalized holomorphic motion of $\alpha^{-1}(A)$.

EXAMPLE 2.11. Let $A = \{0, 1, \infty, a\}$ and $p : \mathbb{D} \to \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ be the holomorphic universal covering map which satisfies $p(0) = a$. Then $\{\varphi_t\}_{t \in \mathbb{D}}$ defined by

$$\varphi_t(0) = 0 \quad \varphi_t(1) = 1 \quad \varphi_t(\infty) = \infty \quad \varphi_t(a) = p(t)$$

is a holomorphic motion of A over $\mathbb{D}$.

EXAMPLE 2.12. Let A be the lattice $\mathbb{Z} \oplus i\mathbb{Z}$ of Gaussian integers and define $\{\varphi_t\}_{t \in \mathbb{H}}$ by

$$\varphi_t(m + in) = m + tn$$

is a holomorphic motion of A over $(\mathbb{H}, i)$.

EXAMPLE 2.13. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasiconformal homeomorphism and $\mu = \mu_f$. For $|t| < 1$, let $\varphi_t = \varphi^{[t\mu]}$ be the normalized solution of the Beltrami equation given by Theorem 1.14. Then φ_t is a holomorphic motion of $\hat{\mathbb{C}}$ over $\mathbb{D}$. This shows that every quasiconformal homeomorphism of the sphere canonically produces a holomorphic motion of the sphere.

EXAMPLE 2.14. Let $U \subset \mathbb{C}$ be a Jordan domain. Suppose there are conformal isomorphisms $f_t^i : U \to U_t^i$ $(i = 0, 1)$ depending holomorphically on a parameter $t \in \mathbb{D}$ such that $\overline{U_t^0}$ and $\overline{U_t^1}$ are disjoint subsets of U. For every finite word $i_1 \cdots i_n$ of 0's and 1's, let

$$U_t^{i_1 \cdots i_n} = f_t^{i_n} \circ \cdots \circ f_t^{i_1}(U)$$

and define the Cantor sets

$$K_t = \bigcap_{n \geq 1} \bigcup_{i_1 \cdots i_n} U_t^{i_1 \cdots i_n}.$$

The K_t determine a holomorphic motion of the base Cantor set K_0 over $\mathbb{D}$. To see this, take a $z \in K_0$ and suppose that it is represented by the infinite word $i_1 i_2 i_3 \ldots$ so that

$$z = U_0^{i_1} \cap U_0^{i_1 i_2} \cap U_0^{i_1 i_2 i_3} \cap \cdots$$

Define

$$\varphi(t, z) = U_t^{i_1} \cap U_t^{i_1 i_2} \cap U_t^{i_1 i_2 i_3} \cap \cdots \in K_t.$$

Note that $\varphi(t, z)$ is the local uniform limit of the sequence of holomorphic functions $\varphi_n(t) = f_t^{i_n} \circ \cdots \circ f_t^{i_1}$, so it depends holomorphically on t. It is now easy to check that $(t, z) \mapsto \varphi(t, z)$ is a holomorphic motion of K_0 over $\mathbb{D}$.

Let $E \subset \hat{\mathbb{C}}$ be a set with at least 4 points. Let us say that a homeomorphism $f : E \to f(E) \subset \hat{\mathbb{C}}$ is ***quasiconformal*** if there exists an $M > 0$ such that

$$\mathrm{dist}_{\hat{\mathbb{C}} \setminus \{0,1,\infty\}} \Big([f(z_1), f(z_2), f(z_3), f(z_4)], [z_1, z_2, z_3, z_4] \Big) \leq M$$

for all quadruples (z_1, z_2, z_3, z_4) in E. Here $\mathrm{dist}_{\hat{\mathbb{C}} \smallsetminus \{0,1,\infty\}}$ is the hyperbolic distance in the thrice punctured sphere and $[z_1, z_2, z_3, z_4]$ is the ***cross ratio*** defined by

$$[z_1, z_2, z_3, z_4] = \frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z_4 - z_2}{z_4 - z_3}.$$

It can be shown that when $E = \hat{\mathbb{C}}$, this definition of quasiconformality agrees with the standard one given in §1.2.

THEOREM 2.15 (The λ-Lemma of Mañe-Sad-Sullivan and Lyubich)**.** *A holomorphic motion $\varphi : \mathbb{D} \times A \to \hat{\mathbb{C}}$ extends uniquely to a holomorphic motion $\Phi : \mathbb{D} \times \overline{A} \to \hat{\mathbb{C}}$. Moreover, Φ is continuous on $\mathbb{D} \times \overline{A}$ and $\Phi_t : \overline{A} \to \Phi_t(\overline{A})$ is a quasiconformal homeomorphism for each $t \in \mathbb{D}$.*

The analogue of the λ-Lemma for continuous motions is certainly false. As an example, let $A = \{1/k\}_{k \geq 1}$ and define the continuous motion $\varphi : \mathbb{R} \times A \to \hat{\mathbb{C}}$ by $\varphi(t, 1/k) = 1/k + ikt$. Evidently, φ has no continuous extension to $\mathbb{R} \times \overline{A}$.

PROOF. Without losing generality, assume that the motion is normalized. By Montel's theorem,

$$\mathcal{F} = \{t \mapsto \varphi(t, z) : z \in A\}$$

is a normal family of holomorphic functions $\mathbb{D} \to \hat{\mathbb{C}}$, so it has compact closure $\overline{\mathcal{F}}$ in $\mathrm{Hol}(\mathbb{D}, \hat{\mathbb{C}})$. Moreover, if $f, g \in \overline{\mathcal{F}}$ are distinct, then $f(t) \neq g(t)$ for all $t \in \mathbb{D}$. To see this, take $f_n, g_n \in \mathcal{F}$ such that $f_n \neq g_n$, $f_n \to f$ and $g_n \to g$, and note that $t \mapsto f_n(t) - g_n(t)$ is nowhere vanishing by the injectivity property of holomorphic motions. It follows from Hurwitz's theorem that $t \mapsto f(t) - g(t)$ is nowhere vanishing as well.

For each $t \in \mathbb{D}$ consider the continuous map

$$p_t : \overline{\mathcal{F}} \to \hat{\mathbb{C}} \qquad p_t(f) = f(t).$$

By the above observation, p_t is injective. Since $\overline{\mathcal{F}}$ is compact, it follows that p_t is a homeomorphism onto its image, which is easily seen to be the closure of $\varphi_t(A)$. Now

$$\Phi(t, z) = p_t \circ p_0^{-1}(z) \qquad (t, z) \in \mathbb{D} \times \overline{A},$$

extends φ to a motion of $\overline{A}$.

The definition of the compact-open topology on $\overline{\mathcal{F}}$ shows that for each $r < 1$, the family $\{p_t\}_{|t| \leq r}$ is equicontinuous, so the same must be true of the family $\{\Phi_t\}_{|t| \leq r}$. It follows that Φ is continuous on the product $\mathbb{D} \times \overline{A}$.

Finally, choose a quadruple (z_1, z_2, z_3, z_4) in $\overline{A}$ and define a holomorphic map $g : \mathbb{D} \to \hat{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$ by

$$g(t) = [\Phi_t(z_1), \Phi_t(z_2), \Phi_t(z_3), \Phi_t(z_4)].$$

By the Schwarz lemma,

$$\mathrm{dist}_{\hat{\mathbb{C}} \smallsetminus \{0,1,\infty\}}(g(t), g(0)) \leq \mathrm{dist}_{\mathbb{D}}(t, 0),$$

which implies that the distance in $\hat{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$ between the cross ratios $[\Phi_t(z_1), \Phi_t(z_2), \Phi_t(z_3), \Phi_t(z_4)]$ and $[z_1, z_2, z_3, z_4)]$ is at most

$$\log\left(\frac{1 + |t|}{1 - |t|}\right).$$

This proves that each $\Phi_t : \overline{A} \to \Phi_t(\overline{A})$ is quasiconformal. □

THEOREM 2.16 (The Improved λ-Lemma of Slodkowski)**.** *A holomorphic motion $\varphi : \mathbb{D} \times A \to \hat{\mathbb{C}}$ extends to a holomorphic motion $\Phi : \mathbb{D} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The extended motion Φ is continuous on $\mathbb{D} \times \hat{\mathbb{C}}$ and $\Phi_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is K_t-quasiconformal for each $t \in \mathbb{D}$, where $K_t = \frac{1+|t|}{1-|t|}$.*

It was proved by Sullivan and Thurston that there exists a universal constant $0 < a < 1$ such that every holomorphic motion of A over $\mathbb{D}$ extends to a holomorphic motion of the sphere over the smaller disk $\mathbb{D}(0, a)$. Bers and Royden proved that one can take $a = 1/3$. Moreover, their extended motion over $\mathbb{D}(0, 1/3)$ has the advantage of being canonical in the sense that the Beltrami form μ_{Φ_t} is harmonic in each component of $\hat{\mathbb{C}} \smallsetminus \overline{A}$. (A Beltrami form μ on a hyperbolic Riemann surface X is called ***harmonic*** if $\mu = \overline{\phi}/\rho^2$, where ϕ is a holomorphic quadratic differential on X and ρ is the density of the hyperbolic metric.) With this additional property, they proved that the extended motion is unique.

As Sullivan and Thurston observed, to obtain the improved λ-Lemma, it suffices to prove the following ***holomorphic axiom of choice***: Given a finite set A and a point $a \notin A$, every holomorphic motion of A over $\mathbb{D}$ extends to a holomorphic motion of $A \cup \{a\}$ over $\mathbb{D}$.

In the original version of the λ-Lemma, $\mathbb{D}$ can be replaced with an arbitrary connected complex manifold, as essentially the same proof works. In the Bers-Royden version, $\mathbb{D}$ can be replaced with the unit ball in any complex normed linear space. In Slodkowski's improved λ-Lemma, $\mathbb{D}$ cannot be replaced for free, as the next example shows.

EXAMPLE 2.17. (Douady) Let $T = \hat{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$, with the base point $t_0 = 2$. Let $A = \{0, 1, 2, \infty\}$ and define a holomorphic motion $\varphi : T \times A \to \hat{\mathbb{C}}$ by

$$\varphi_t(0) = 0 \quad \varphi_t(1) = 1 \quad \varphi_t(\infty) = \infty \quad \varphi_t(2) = t.$$

This motion is *maximal* in the sense that it cannot be extended to a holomorphic motion of any larger set over T. To see this, it suffices to show that every holomorphic map $f : T \to T$ has a fixed point. Assume by way of contradiction that such a fixed point free map exists. By Picard's great theorem, none of $0, 1, \infty$ can be an essential singularity for f, so f extends to a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 1$. As $f^{-1}\{0, 1, \infty\} \subset \{0, 1, \infty\}$, f acts bijectively on $\{0, 1, \infty\}$. By the assumption all the $d + 1$ fixed points of f belong to $\{0, 1, \infty\}$. If $d = 1$, f is a Möbius map which fixes one of $0, 1, \infty$ with multiplicity 2 and swaps the other two. But any Möbius map with a double fixed point is conjugate to $z \mapsto z + 1$ so it cannot swap a pair of distinct points. If $d \geq 2$, each fixed point in $\{0, 1, \infty\}$ is a critical point of order $d - 1$ and in particular is a simple (i.e., multiplicity 1) fixed point. This forces $d + 1 \leq 3$, or $d = 2$. But then each of $0, 1, \infty$ is a simple fixed point of f as well as a critical point. This is impossible since a degree 2 rational map has only 2 critical points.

Problems.

(1) Let $f \in \mathrm{Rat}_d$ have an attracting fixed point p_0 with multiplier λ_0. Show that for every $\lambda \in \mathbb{D}^*$ there exists $f_\lambda \in \mathrm{Rat}_d$, quasiconformally conjugate to f, with an attracting fixed point p_λ of multiplier λ.