QUATERNIONS AND ROTATIONS

S. ZAKERI, 8-11-22

The universal covering map $SU(2) \to SO(3)$ is ubiquitous in mathematics and physics and can be understood mechanically from the isomorphic covering $S^3 \to \mathbb{R}P^3$ given by identifying pairs of antipodal points. This expository note will describe a known alternative view of this covering map using quaternions, which shows it is essentially inherited from the angle-doubling map of the circle.[1]

§ 1. Quaternions. Recall that a ***quaternion*** is a symbol of the form

$$q = t + xi + yj + zk \tag{1}$$

where $t, x, y, z \in \mathbb{R}$ and the units i, j, k satisfy the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j.$$

With the obvious addition and multiplication generated by the above relations, quaternions form a division ring (i.e., a "non-commutative field") that is traditionally denoted by $\mathbb{H}$.

The ***conjugate*** of q in (1) is defined by

$$q^* = t - xi - yj - zk.$$

Conjugation satisfies the basic property

$$(q_1 q_2)^* = q_2^* q_1^*.$$

The ***real part*** and ***vector part*** of q are defined by

$$\begin{aligned} \mathrm{Re}(q) &= \tfrac{1}{2}(q + q^*) = t \\ \mathrm{Ve}(q) &= \tfrac{1}{2}(q - q^*) = xi + yj + zk. \end{aligned}$$

Thus, for every $q \in \mathbb{H}$ we have the unique decompositions

$$q = \mathrm{Re}(q) + \mathrm{Ve}(q) \quad \text{and} \quad q^* = \mathrm{Re}(q) - \mathrm{Ve}(q).$$

The ***norm*** of q is the non-negative number

$$\|q\| = (qq^*)^{1/2} = (t^2 + x^2 + y^2 + z^2)^{1/2}.$$

[1]The century old idea of representing rotations by quaternions has been revitalized in our time by applications in modern computer graphics.

It is multiplicative in the sense that

$$\|q_1 q_2\| = \|q_1\| \, \|q_2\|.$$

Every non-zero $q \in \mathbb{H}$ has a multiplicative inverse q^{-1} given by

$$q^{-1} = \frac{q^*}{\|q\|^2}.$$

§ 2. Vectors. Consider the additive subgroup $\mathbb{H}_0 \subset \mathbb{H}$ consisting of all quaternions with vanishing real part:

$$\mathbb{H}_0 = \{q \in \mathbb{H} : \mathrm{Re}(q) = 0\} = \{q \in \mathbb{H} : q^* = -q\}.$$

In other words, $q \in \mathbb{H}_0$ if and only if $q = \mathrm{Ve}(q)$. For this reason, we refer to elements of $\mathbb{H}_0$ as ***vectors***. The terminology is justified since there is a natural isomorphism $\mathbb{H}_0 \to \mathbb{R}^3$ given by

$$xi + yj + zk \mapsto (x, y, z).$$

Under this isomorphism, the norm of a vector $v \in \mathbb{H}_0$ is the same as its Euclidean norm, and

$$\|v\|^2 = vv^* = -v^2.$$

In particular

$$v^2 = -1 \qquad \text{for every unit vector } v. \tag{2}$$

A brief computation shows that the Euclidean scalar and cross products on $\mathbb{R}^3$ have quaternionic expressions

$$\begin{aligned} u \cdot v &= -\mathrm{Re}(uv) = -\tfrac{1}{2}(uv + vu) \\ u \times v &= \;\;\; \mathrm{Ve}(uv) \;=\; \tfrac{1}{2}(uv - vu). \end{aligned}$$

In particular,

$$u, v \text{ are orthogonal} \iff uv = -vu \tag{3}$$

$$u, v \text{ are parallel} \iff uv = vu. \tag{4}$$

Thus, the quaternion product of vectors in $\mathbb{H}_0$ beautifully records both the scalar and cross products:

$$uv = \underbrace{-u \cdot v}_{\text{real part}} + \underbrace{u \times v}_{\text{vector part}}$$

Observe that $\mathbb{H}_0$ is not closed under quaternion multiplication. In fact, if $u, v \in \mathbb{H}_0$, then $uv \in \mathbb{H}_0$ if and only if u, v are orthogonal.

§ 3. Unit quaternions. Consider now the non-abelian multiplicative subgroup $\mathbb{H}_1 \subset \mathbb{H} \setminus \{0\}$ consisting of all quaternions with norm 1:

$$\mathbb{H}_1 = \{q \in \mathbb{H} : \|q\| = 1\} = \{q \in \mathbb{H} : q^* = q^{-1}\}.$$

Topologically $\mathbb{H}_1$ is homeomorphic to the unit 3-sphere

$$S^3 = \{(t, x, y, z) \in \mathbb{R}^4 : t^2 + x^2 + y^2 + z^2 = 1\}$$

(indeed, this is the easiest way of showing that S^3 admits the structure of a Lie group). The following result will be used in the next section:

Lemma 1. *Every $q \in \mathbb{H}_1$ can be written as*

$$q = \cos\theta + u \sin\theta,$$

where $u \in \mathbb{H}_0 \cap \mathbb{H}_1$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$. If $q = 1$ or -1, then $\theta = 0$ or $\pi \pmod{2\pi\mathbb{Z}}$ but u is completely arbitrary. If $q \neq \pm 1$, then the pair (u, θ) is uniquely determined up to the sign change $(-u, -\theta)$.

The proof is an easy exercise.

Elements of $\mathbb{H}_1$ can also be represented by 2×2 complex matrices as follows. Every $q = t + xi + yj + zk \in \mathbb{H}_1$ can be written as $q = a + bj$, where $a = t + xi$ and $b = y + zi$ are a pair of complex numbers with $|a|^2 + |b|^2 = \|q\|^2 = 1$. The quaternion multiplication of $q_1 = a + bj$ and $q_2 = c + dj$ turns into

$$\begin{aligned} q_1 q_2 &= ac + adj + bjc + bjdj = ac + adj + b\bar{c}j - b\bar{d} \\ &= (ac - b\bar{d}) + (ad + b\bar{c})j. \end{aligned}$$

This suggests the representation of $q = a + bj \in \mathbb{H}_1$ as the matrix

$$\underline{q} = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

and shows that the map $\mathbb{H}_1 \to SU(2)$ defined by $q \mapsto \underline{q}$ is a group isomorphism. Note that under this isomorphism the quaternion units $1, i, j, k$ map to

$$\underline{1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad \underline{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \underline{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Moreover, the conjugate q^* corresponds to the adjoint matrix $\underline{q}^*$ (i.e., the conjugate transpose of $\underline{q}$).

§ 4. Rigid rotations of $\mathbb{H}_0 \cong \mathbb{R}^3$.

Lemma 2. *For every* $q \in \mathbb{H}_1$ *the conjugation* $T_q : v \mapsto qvq^{-1}$ *acts on* $\mathbb{H}_0$*. Under the natural isomorphism* $\mathbb{H}_0 \cong \mathbb{R}^3$, T_q *is a rigid rotation about the origin.*

Proof. If $v \in \mathbb{H}_0$, then

$$(qvq^{-1})^* = (q^{-1})^* v^* q^* = -qvq^{-1},$$

so $qvq^{-1} \in \mathbb{H}_0$. Thus, T_q maps $\mathbb{H}_0$ to itself. Moreover,

$$\|qvq^{-1}\| = \|q\| \, \|v\| \, \|q^{-1}\| = \|v\|,$$

so T_q is an isometry of $\mathbb{R}^3$ fixing the origin, i.e., an element of $O(3)$. Since $T_1 = \mathrm{id}$ and since $q \mapsto T_q$ is continuous on the connected space $\mathbb{H}_1 \cong S^3$, we conclude that T_q is orientation-preserving for every $q \in \mathbb{H}_1$, so $T_q \in SO(3)$. □

Lemma 3. *Let* $q \in \mathbb{H}_1$ *and consider the representation* $q = \cos\theta + u \sin\theta$ *as in Lemma 1. Then* T_q *is the rotation of* $\mathbb{R}^3$ *by the angle* 2θ *about the axis* u*.*

Notice that in this statement the ambiguity of the pair (u, θ) is absorbed by the ambiguity of the axis and angle of a rotation. In fact, if $q \neq \pm 1$, then the pair (u, θ) is uniquely determined up to the sign $\pm$, consistent with the fact that the rotation of $\mathbb{R}^3$ by 2θ about u is the same as the rotation by -2θ about $-u$. On the other hand, if $q = 1$ or -1, then $\theta = 0$ or $\pi \pmod{2\pi\mathbb{Z}}$ and u is arbitrary, but again the ambiguity is not an issue because in this case $T_q = \mathrm{id}$.

Proof. It suffices to assume $q \neq \pm 1$. Note that since $q \in \mathbb{H}_1$,

$$q^{-1} = q^* = \cos\theta - u \sin\theta.$$

If v is parallel to u, then

$$\begin{aligned} T_q(v) = qvq^{-1} = qvq^* &= (\cos\theta + u\sin\theta)\, v\, (\cos\theta - u\sin\theta) \\ &= v\cos^2\theta + (uv - vu)\cos\theta\sin\theta - uvu\sin^2\theta \\ &= v\,(\cos^2\theta - u^2\sin^2\theta) && \text{(by(4))} \\ &= v\,(\cos^2\theta + \sin^2\theta) = v && \text{(by(2))} \end{aligned}$$

Suppose now that v is a unit vector orthogonal to u, so $uv = -vu$ by (3). Set $w = u \times v = -vu$. Then the triple (v, w, u) forms a positive orthonormal basis for $\mathbb{H}_0 \cong \mathbb{R}^3$. The computation

$$\begin{aligned} T_q(v) &= v\cos^2\theta + (uv - vu)\cos\theta\sin\theta - uvu\sin^2\theta \\ &= v\cos^2\theta + 2w\cos\theta\sin\theta + vu^2\sin^2\theta \\ &= v\cos^2\theta - v\sin^2\theta + 2w\cos\theta\sin\theta \\ &= v\cos(2\theta) + w\sin(2\theta) \end{aligned}$$

shows that $T_q(v)$ is the rotation of v in the plane $u^\perp$ by the angle 2θ. □

Corollary 4. *The map* $\Phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ *defined by* $\Phi(\underline{q}) = T_q$ *is a surjective homomorphism with kernel* $\{\pm I\}$*:*

$$1 \mapsto \{\pm I\} \to \mathrm{SU}(2) \to \mathrm{SO}(3) \to 1.$$

Topologically, Φ *is a degree 2 covering map.*

Proof. Φ is a homomorphism since

$$T_{q_1 q_2}(v) = (q_1 q_2) v (q_1 q_2)^* = q_1 (q_2 v q_2^*) q_1^* = T_{q_1}(T_{q_2}(v)),$$

and it is surjective by Lemma 3. The condition $\Phi(\underline{q}) = \mathrm{id}$ means $qv = vq$ for every $v \in \mathbb{H}_0$. Writing $q = t + u$ where $t = \mathrm{Re}(q) \in \mathbb{R}, u = \mathrm{Ve}(q) \in \mathbb{H}_0$, this condition translates to $uv = vu$ or $u \times v = 0$ for every $v \in \mathbb{H}_0$. Thus $u = 0$ and $q = t$. Since $\|q\| = |t| = 1$, we obtain $q = \pm 1$.

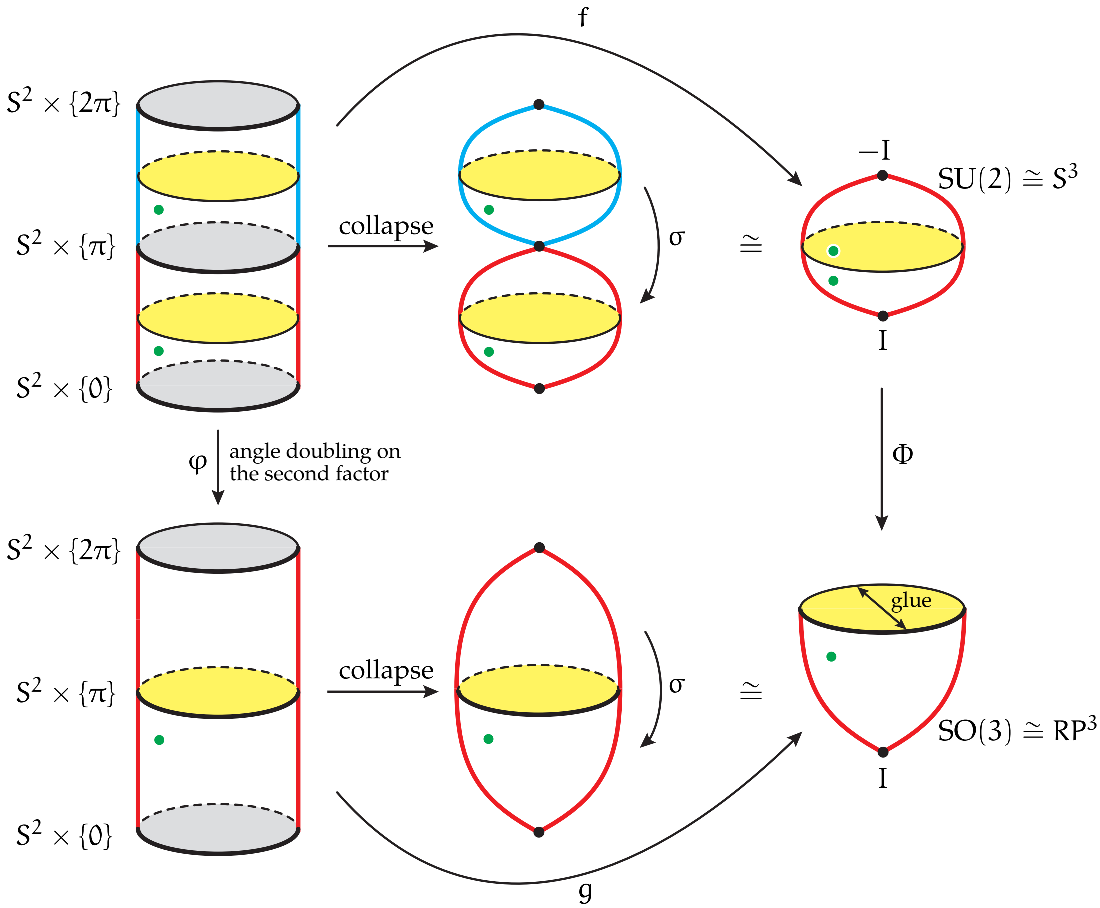
If $\Phi(\underline{q}) = T$, then $\Phi^{-1}(T) = \pm \underline{q}$ and there is a small neighborhood U of $\underline{q}$ which is disjoint from $-U$. It follows that Φ is a local homeomorphism. Since $\mathrm{SU}(2)$ is compact, Φ is automatically proper, and we conclude that Φ is a covering map of degree 2. □

§ 5. Geometric interpretation of Φ. The topology of the covering map $\Phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is usually explained by considering homeomorphisms $\mathrm{SU}(2) \cong S^3$, $\mathrm{SO}(3) \cong \mathrm{RP}^3$ and looking at the isomorphic covering map $S^3 \to \mathrm{RP}^3$ obtained by identifying antipodal pairs. A (perhaps superficially) alternative view is given by the above quaternionic description which reveals more vividly that the 2-to-1 nature of Φ is due to the angle doubling map of the circle. Here is how:

For simplicity let us identify $\mathbb{H}_0 \cap \mathbb{H}_1$ with the unit sphere $S^2 \subset \mathbb{R}^3$ and use the notation $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Consider the surjection $f : S^2 \times S^1 \to \mathrm{SU}(2)$ given by $f(u, \theta) = \underline{q}$, where $q = \cos\theta + u \sin\theta$ as in Lemma 1. Evidently $\mathrm{SU}(2)$ can be identified with the quotient space $(S^2 \times S^1)/f$. Note that f collapses the horizontal sections $S^2 \times \{0\}$ and $S^2 \times \{\pi\}$ to the points $\pm I$ and maps all other sections homeomorphically onto their images. Moreover, $f = f \circ \sigma$, where σ is the involution $(u, \theta) \mapsto (-u, -\theta)$. This shows that $\mathrm{SU}(2)$ is obtained topologically from $S^2 \times S^1$ by collapsing the sections $S^2 \times \{0\}, S^2 \times \{\pi\}$ and then taking the quotient under σ (see the figure).

Similarly, consider the surjection $g : S^2 \times S^1 \to \mathrm{SO}(3)$, where $g(u, \theta)$ is the rotation of $\mathbb{R}^3$ by the angle θ about the axis u. Then $\mathrm{SO}(3)$ is homeomorphic to the quotient space $(S^2 \times S^1)/g$. Since g collapses the horizontal section $S^2 \times \{0\}$ to the point I and satisfies $g = g \circ \sigma$, it follows that $\mathrm{SO}(3)$ is obtained from $S^2 \times S^1$ by collapsing the section $S^2 \times \{0\}$ and then taking the quotient under σ.

Now the degree 2 covering map $\varphi : S^2 \times S^1 \to S^2 \times S^1$ defined by $(u, \theta) \mapsto (u, 2\theta)$ commutes with σ and sends the fibers of f to the fibers of g, so it descends to a degree 2 covering map $SU(2) \to SO(3)$ which, by Lemma 3, is precisely Φ:



The figure shows that the trivial horizontal foliation of $S^2 \times S^1$ induces a foliation $\mathcal{F}_1$ of $SU(2) - \{\pm I\}$ by 2-spheres (this is just the foliation of $S^3 - \{0, \infty\} = \mathbb{R}^3 - \{0\}$ by concentric spheres centered at the origin). It also induces a foliation $\mathcal{F}_2$ of $SO(3) - \{I\}$ whose leaves are all 2-spheres except for a single leaf homeomorphic to RP^2 (the one corresponding to rotations by the angle π, shown in yellow). The covering map Φ carries $\mathcal{F}_1$ onto $\mathcal{F}_2$. More precisely, every spherical leaf of $\mathcal{F}_2$ lifts under Φ to the disjoint union of two spherical leaves of $\mathcal{F}_1$ on which Φ acts homeomorphically. The single leaf of $\mathcal{F}_2$ homeomorphic to RP^2 lifts to a single spherical leaf of $\mathcal{F}_1$ on which Φ acts as a degree 2 covering.

It is instructive to compare this situation with the following 1-dimensional toy model: Let $\sigma : S^1 \to S^1$ be the involution $\sigma(\theta) = -\theta \pmod{2\pi\mathbb{Z}}$. The doubling map $\varphi : S^1 \to S^1$ defined by $\varphi(\theta) = 2\theta \pmod{2\pi\mathbb{Z}}$ commutes with σ, so it induces a well-defined map $\Phi : S^1/\sigma \to S^1/\sigma$ *which is no longer a covering*. In fact, the map $\theta \mapsto x = \cos\theta$ induces a homeomorphism $S^1/\sigma \to [-1, 1]$ under which Φ will be conjugate to the Chebyshev polynomial $x \mapsto 2x^2 - 1$, which nearly misses being a 2-to-1 covering due to the presence of the critical point at $x = 0$.

Department of Mathematics, Queens College of CUNY, 65-30 Kissena Blvd., Queens, New York 11367, USA

The Graduate Center of CUNY, 365 Fifth Ave., New York, NY 10016, USA
E-mail address: saeed.zakeri@qc.cuny.edu