SULLIVAN’S PROOF OF FATOU’S NO WANDERING DOMAIN
CONJECTURE

S. ZAKERI

Abstract. A self-contained and simplified version of Sullivan’s proof, following N.
Baker and C. McMullen.

§1. Set up. Let \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a rational map of degree \( d \geq 2 \). Let \( J(f) \) and \( F(f) \) denote the Julia set and the Fatou set of \( f \), respectively. Recall that the open
set \( F(f) \) consists of all points near which the family of iterates \( \{f^n\} \) is normal, and
\( J(f) = \hat{\mathbb{C}} \setminus F(f) \). The Julia set also coincides with the closure of the set of repelling
periodic points of \( f \). Every connected component of \( F(f) \) is called a Fatou component. The image \( f(U) \) of a Fatou component \( U \) is itself a Fatou component and the
mapping \( f : U \to f(U) \) is proper of some degree \( \leq d \).

Theorem (Sullivan). Every Fatou component \( U \) of \( f \) is eventually periodic, that is,
there exist \( n > m > 0 \) such that \( f^n(U) = f^m(U) \).

The idea of the proof is as follows: Assuming there exists a wandering Fatou com-
ponent \( U \) (or simply a wandering domain), we change the conformal structure of the
sphere along the grand orbit of \( U \) to find an infinite-dimensional family of rational
maps of degree \( d \), all quasiconformally conjugate to \( f \). This is a contradiction since
the space \( \text{Rat}_d \) of rational maps of degree \( d \), as a Zariski open subset of \( \mathbb{C}P^{2d+1} \), is
finite-dimensional.

Remark. The corresponding statement for entire maps is false. For example, the
map \( z \mapsto z + \sin(2\pi z) \) has wandering domains.

§2. A reduction. The following observation drastically simplifies part of Sullivan’s
original argument.

Lemma (Baker). If \( U \) is a wandering domain, then \( f^n(U) \) is simply-connected for
all large \( n \).

Proof. Let \( U_n = f^n(U) \). Replacing \( U \) by \( U_k \) for some large \( k \) if necessary, we may
assume that no \( U_n \) contains a critical point of \( f \), so that \( f^n : U \to U_n \) is a covering
map for all \( n \). We can also arrange that \( \infty \in U \). Since the \( U_n \) are disjoint subsets of
\( \mathbb{C} \setminus U \) for \( n \geq 1 \), we have \( \text{area}(U_n) \to 0 \). But \( \{f^n|_U\} \) is a normal family, so every
convergent subsequence of this sequence must be a constant function. In particular, \( \text{diam}(f^{on}(K)) \to 0 \) for every compact set \( K \subset U \).

Now take any loop \( \gamma \subset U \) and set \( \gamma_n = f^{on}(\gamma) \subset U_n \). By the above argument \( \text{diam}(\gamma_n) \to 0 \). If \( B_n \) is the union of the bounded components of \( \mathbb{C} \setminus \gamma_n \), it follows that \( \text{diam}(B_n) \to 0 \) also. Since \( f(B_n) \) is open, \( \partial f(B_n) \subset \gamma_{n+1} \), and \( \text{diam}(f(B_n)) \to 0 \), we must have \( f(B_n) \subset \overline{B_{n+1}} \) for large \( n \). In particular, the iterated images of \( B_n \) are subsets of \( \mathbb{C} \setminus U \) for large \( n \). Montel’s theorem then implies \( B_n \subset F(f) \), which gives \( B_n \subset U_n \). Thus \( \gamma_n \) is null-homotopic in \( U_n \) for large \( n \). Since \( f^{on} : U \to U_n \) is a covering map, we can lift this homotopy to \( U \). This proves that \( U \) is simply connected.

\[ \square \]

§3. Constructing deformations. Let \( f \) have a wandering domain \( U \). In view of the above lemma, we can assume that \( U_n = f^{on}(U) \) is simply-connected and \( f : U_n \to U_{n+1} \) is a conformal isomorphism for all \( n \geq 0 \). Given an \( L^\infty \) Beltrami differential \( \mu \) defined on \( U \), we can construct an \( f \)-invariant \( L^\infty \) Beltrami differential on \( \hat{\mathbb{C}} \) as follows. Use the forward and backward iterates of \( f \) to spread \( \mu \) along the grand orbit

\[ \text{GO}(U) = \{ z \in \hat{\mathbb{C}} : f^{on}(z) \in U_m \text{ for some } n, m \geq 0 \}. \]

On the complement \( \hat{\mathbb{C}} \setminus \text{GO}(U) \), set \( \mu = 0 \). The resulting Beltrami differential is defined almost everywhere on \( \hat{\mathbb{C}} \), it satisfies \( f^* \mu = \mu \) by the way it is defined, and \( \| \mu \|_{\infty} < \infty \) since spreading \( \mu|_U \) along \( \text{GO}(U) \) by the iterates of the holomorphic map \( f \) does not change the dilatation. Now consider the deformation \( \mu_t = t\mu \) for \( |t| < \varepsilon \), where \( \varepsilon > 0 \) is small enough to guarantee \( \| \mu_t \|_{\infty} < 1 \) if \( |t| < \varepsilon \). Note that since \( f \) is holomorphic, \( f^* \) acts as a linear rotation, so \( f^* \mu_t = \mu_t \). Let \( \varphi_t = \varphi^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be the normalized solution of the Beltrami equation \( \overline{\partial} \varphi_t = \mu_t \overline{\partial} \varphi_t \) which fixes \( 0, 1, \infty \).

It is easy to see that \( f_t = \varphi_t \circ f \circ \varphi_t^{-1} \) is a rational map of degree \( d \), and \( t \mapsto f_t \) is holomorphic, with \( f_0 = f \). The infinitesimal variation

\[ w(z) = \frac{d}{dt} \bigg|_{t=0} f_t(z) \]

defines a holomorphic vector field whose value at \( z \) lies in the tangent space \( T_{f(z)} \hat{\mathbb{C}} \). In other words, \( w \) can be thought of as a holomorphic section of the pull-back bundle \( f^*(T\hat{\mathbb{C}}) \) which in turn can be identified with a tangent vector in \( T_{f} \text{Rat}_d \). This is the so-called infinitesimal deformation of \( f \) induced by \( \mu \). We say that \( \mu \) induces a trivial deformation if \( w = 0 \).

Another way of describing \( w \) is as follows: First consider the unique quasiconformal vector field solution to the \( \partial \overline{\partial} \)-equation \( \overline{\partial} v = \mu \) which vanishes at \( 0, 1, \infty \). This is precisely the infinitesimal variation \( \frac{d}{dt} \bigg|_{t=0} \varphi_t(z) \) of the normalized solution of the Beltrami equation. It is not hard to check that \( w = \delta_f v \), where

\[ \delta_f v(z) = v(f(z)) - f'(z)v(z) \]
measures the deviation of $v$ from being $f$-invariant. Note in particular that $\delta_f v$ is holomorphic even though $v$ is only quasiconformal, and that $w = \delta_f v$ depends linearly on $\mu$, a fact that is not immediately clear from the first description of $w$. It follows that $\mu$ induces a trivial deformation if and only if $v$ is $f$-invariant.

It is easy to see that the triviality condition $\delta_f v = 0$ forces $v$ to vanish on the Julia set $J(f)$. In fact, let $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$ be a repelling cycle of $f$ with multiplier $\lambda$. Then the condition $\delta_f v = 0$ implies $v(z_{j+1}) = f'(z_j)v(z_j)$ for all $j = 0, \ldots, n-1$, so that

$$\prod_{j=0}^{n-1} v(z_j) = \lambda \cdot \prod_{j=0}^{n-1} v(z_j).$$

Since $|\lambda| > 1$, it follows that $v(z_j) = 0$ for some, hence for all $j$. Now $J(f)$ is the closure of such cycles and $v$ is continuous, so $v(z) = 0$ for all $z \in J(f)$.

**§4. The proof.** The above construction gives well-defined linear maps

$$B(U) \stackrel{\lambda}{\rightarrow} B(\hat{\mathbb{C}}, f) \stackrel{D}{\rightarrow} T_f \text{Rat}_d$$

Here $B(U)$ is the space of $L^\infty$ Beltrami differentials in $U$, $B(\hat{\mathbb{C}}, f)$ is the space of $f$-invariant $L^\infty$ Beltrami differentials on $\hat{\mathbb{C}}$, and $D$ is the linear operator $D\mu = w = \delta_f v$ constructed above.

**Lemma.** $B(U)$ contains an infinite-dimensional subspace $N(U)$ of compactly supported Beltrami differentials with the following property: If $\mu N(U)$ satisfies $\mu = \overline{\partial} v$ for some quasiconformal vector field $v$ with $v|_{\partial U} = 0$, then $\mu = 0$.

Assuming this for a moment, let us see how this implies the theorem. Consider the above subspace $N(U)$ for a simply-connected wandering domain $U$ and restrict the diagram (1) to this subspace. If $D(\mu) = 0$ for some $\mu \in N(U)$, or in other words if $\mu$ induces a trivial deformation, that means the normalized solution $v$ to $\overline{\partial} v = \mu$ is $f$-invariant. Hence $v = 0$ on $J(f)$ and in particular on the boundary of $U$. By the property of $N(U)$, $\mu = 0$. This means that the infinite-dimensional subspace $N(U)$ injects into $T_f \text{Rat}_d$ whose dimension is $2d + 1$. The contradiction shows that no wandering domain can exist.

It remains to prove the Lemma. Let us first consider the corresponding problem for the unit disk $\mathbb{D}$. Let $\hat{N}(\mathbb{D}) \subset B(\mathbb{D})$ be the linear span of the Beltrami differentials $\mu_k(z) = z^{k+1} \frac{dz}{dz}$ for $k \geq 0$. The vector field

$$V_k(z) = \begin{cases} 
\frac{1}{k+1} z^{k+1} \frac{\partial}{\partial z} & |z| < 1 \\
\frac{1}{k+1} z^{-(k+1)} \frac{\partial}{\partial z} & |z| \geq 1
\end{cases}$$

solves the equation $\overline{\partial} V_k = \mu_k$ on $\mathbb{D}$. Let $\mu = \overline{\partial} v \in \hat{N}(\mathbb{D})$ and $v|_{\partial \mathbb{D}} = 0$, and take the appropriate linear combination $V$ of the $V_k$ which solves $\overline{\partial} V = \mu$. Then $V - v$ is
holomorphic in \( \mathbb{D} \) and coincides with \( V \) on the boundary \( \partial \mathbb{D} \). This is impossible if \( V|_{\partial \mathbb{D}} \) has any negative power of \( z \) in it. Hence \( \mu = 0 \). To get the compact support condition, let \( N(\mathbb{D}) \subset B(U) \) consist of all Beltrami differentials which coincide with an element of \( \hat{N}(\mathbb{D}) \) on the disk \( |z| < 1/2 \) and are zero on \( 1/2 \leq |z| < 1 \). If \( \mu = \overline{\partial} v \in N(\mathbb{D}) \) and \( v|_{\partial \mathbb{D}} = 0 \), then \( v \) has to be zero on the annulus \( 1/2 < |z| < 1 \) since it is holomorphic there. In particular, it is zero on \( |z| = 1/2 \). Now the same argument applied to the disk \( |z| < 1/2 \) shows \( \mu = 0 \).

For the general case, consider a conformal isomorphism \( \psi : \mathbb{D} \to U \) with the inverse \( \phi = \psi^{-1} \) and define \( N(U) = \phi^*(N(\mathbb{D})) \). Let \( v = v(z) \frac{\partial}{\partial z} \) be a quasiconformal vector field such that \( \mu = \overline{\partial} v \in N(U) \) and \( v|_{\partial U} = 0 \). Then \( \phi^*(v) = v(\psi(z))/\psi'(z) \frac{\partial}{\partial z} \) is a vector field on \( \mathbb{D} \) which is holomorphic near the boundary \( \partial \mathbb{D} \) and \( v(\psi(z)) \to 0 \) as \( |z| \to 1 \). By the reflection principle, \( v(\psi(z)) \) is identically zero near the boundary of \( \mathbb{D} \). Since \( \psi^* \mu = \overline{\partial} \phi^*(v) \in N(\mathbb{D}) \), we must have \( \psi^* \mu = 0 \), which implies \( \mu = 0 \). \( \square \)

**Remark.** Sullivan’s original argument [Ann. of Math. 122 (1985) 401-418] had to deal with two essential difficulties: (i) the possibility of \( U \) being non simply-connected, perhaps of infinite topological type; (ii) the possible complications near the boundary of \( U \), for example when \( \partial U \) is not locally-connected. He addressed the former by using a direct limit argument, and the latter by using Carathéodory’s theory of “prime ends.” Both of these difficulties are surprisingly bypassed in the present proof.