Math 142 – Midterm Exam 3

Directions. Read each question on this exam before you start working so you can get the flavor of the questions. Please show all of your work. Unsupported answers will not even be graded. Do not cheat, else you pay with your academic life.

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1. Find the area of the region bounded by $y = 1/x, y = x^2, y = 0, x = e$.

Solution: Note that the two graphs intersect when $x = 1$. Also notice that the region in question is the region under the graph $y = x^2$ from $x = 0$ to $x = 1$, and the region under the graph $y = 1/x$ from $x = 1$ to $x = e$. Therefore, the area underneath equals

$$
\int_{0}^{1} x^2 dx + \int_{1}^{e} \frac{1}{x} dx = \frac{1}{3} + 1 = \frac{4}{3}.
$$
2. Let \( \mathcal{R} \) be the region in the first quadrant bounded by the curves \( y = x^3 \) and \( y = 2x - x^2 \). Calculate the following quantities:

(a) The area of \( \mathcal{R} \).

Solution: The two curves intersect at \( x = 0, x = -2 \), and \( x = 1 \). The region bounded by the curves occurs from \( x = 0 \) to \( x = 1 \). Moreover, in this region, the graph of \( y = 2x - x^2 \) is above the graph of \( y = x^3 \). Therefore, the area bounded by the curves is:

\[
\int_{0}^{1} [(2x - x^2) - x^3] \, dx = 1 - 1/3 - 1/4 = 5/12.
\]

(b) The volume obtained by rotating \( \mathcal{R} \) around the \( x \)-axis.

Solution: Use cross-sectional area method. Thus the volume is

\[
\int_{0}^{1} \pi[(2x - x^2)^2 - (x^3)^2] \, dx = (4/3 + 1/5 - 1 - 1/7)\pi
\]

(c) The volume obtained by rotating \( \mathcal{R} \) around the \( y \)-axis.

Solution: Use cylindrical shells method. Thus the volume is

\[
\int_{0}^{1} 2\pi x f(x) \, dx = \int_{0}^{1} 2\pi x[(2x - x^2) - x^3] \, dx = (4/3 - 1/2 - 2/5)\pi
\]
3. Find the length of the curve \( y = \frac{1}{6}(x^2 + 4)^{3/2} \) on the interval \( 0 \leq x \leq 3 \).

Solution: \( f(x) = \frac{1}{6}(x^2 + 4)^{3/2} \), so \( f'(x) = \frac{1}{4}(x^2 + 4)^{1/2}(2x) \). Then the arc length is

\[
\int_{0}^{3} \sqrt{1 + f'(x)^2} dx = \int_{0}^{3} \sqrt{1 + \left[ \frac{1}{4}(x^2 + 4)^{1/2}(2x) \right]^2} dx
\]

\[
= \int_{0}^{3} \sqrt{\frac{x^4}{4} + x^2 + 1} dx = \int_{0}^{3} \sqrt{\frac{1}{2}x^2 + 1} dx
\]

\[
= \int_{0}^{3} \left( \frac{1}{2}x^2 + 1 \right) dx = \frac{x^3}{6} + x \bigg|_{0}^{3} = \frac{27}{6} + 3 = 27/6 + 3.
\]
4. Solve the differential equation $2y e^{y^2} y' = 2x + 3\sqrt{x}$.

Solution: The equation is

$$2y e^{y^2} \frac{dy}{dx} = 2x + 3\sqrt{x}.$$ 

This is a separable equation. Move all $x$'s to one side, all $y$'s to the other side, so we get

$$2y e^{y^2} dy = (2x + 3\sqrt{x}) dx.$$ 

We now integrate both sides:

$$\int 2y e^{y^2} dy = \int (2x + 3\sqrt{x}) dx.$$ 

For the left hand side, we need to use u substitution: Set $u = y^2$, so $du/\frac{dy}{y} = 2y$, so that $\int 2y e^{y^2} dy = \int e^u du$. Thus we have

$$\int e^u du = \int (2x + 3\sqrt{x}) dx,$$

which becomes

$$e^u = x^2 + 2x^{3/2} + C,$$

which is the same as $e^{y^2} = x^2 + 2x^{3/2} + C$. Take natural log of both sides gives

$$y^2 = \ln(x^2 + 2x^{3/2} + C),$$

and now taking square root gives

$$y = \sqrt{\ln(x^2 + 2x^{3/2} + C)}.$$
5. Find the volume of the solid obtained by rotating the region bounded by the curves 
\( x = 0 \) and \( x = 9 - y^2 \) around the line \( x = -1 \).

Solution: The curves \( x = 0 \) and \( x = 9 - y^2 \) intersect at \( y = -3, 3 \). Revolving \( x = \) functions around a line parallel to the \( y \)-axis means we should do cross-sectional area. The cross section at \( y \) is a washer/shell, with inner radius 1 (since the \( y \)-axis is distance 1 away from the line \( x = -1 \)), and outer radius \( 9 - y^2 + 1 = 10 - y^2 \). Therefore, we get that the volume equals

\[
\int_{-3}^{3} \pi [(10 - y^2)^2 - 1^2] \, dy = 99y - \frac{20}{3}y^3 + \frac{1}{5}y^5 \bigg|_{-3}^{3} = 594 - 360 + \frac{486}{5} = 234 + \frac{486}{5}.
\]