Degree Spectra of Differentially Closed Fields

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Joint work with Dave Marker.
Spectra of Countable Structures

Let $S$ be a structure with domain $\omega$, in a finite language.

**Definition**

The *Turing degree* of $S$ is the join of the Turing degrees of the functions and relations on $S$. If these are all computable, then $S$ is a *computable structure*.

**Definition**

The *spectrum* of $S$ is the set of all Turing degrees of copies of $S$:

$$\text{Spec}(S) = \{ \deg(M) : M \cong S \& \text{dom}(M) = \omega \}.$$  

So the spectrum measures the level of complexity intrinsic to the structure $S$. 

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Facts About Spectra

**Theorem (Knight 1986)**

For all countable structures $S$ but the automorphically trivial ones, the spectrum of $S$ is upwards-closed under Turing reducibility.

Many interesting spectra can be built using graphs, including upper cones, $\alpha$-th jump cones $\{d : d^{(\alpha)} \geq_T c\}$, and more exotic sets of Turing degrees. (Greenberg, Montalbán, and Slaman recently constructed a graph whose spectrum contains exactly the nonhyperarithmetic degrees.) Indeed, graphs are *complete*, in the following sense:

**Theorem (Hirschfeldt-Khoussainov-Shore-Slinko 2002)**

For every countable structure $S$ in any finite language, there exists a countable graph $G$ which has the same spectrum as $S$. 
Spectra of Algebraically Closed Fields

\{\text{all Turing degrees}\}.
Differentially Closed Fields

A differential field is a field along with a differential operator \( \delta \) on the field elements, respecting addition \( \delta(x + y) = \delta x + \delta y \) and satisfying the product rule \( \delta(x \cdot y) = (x \cdot \delta y) + (y \cdot \delta x) \).

Such a field \( K \) is differentially closed if it also satisfies the Blum axioms: for all differential polynomials \( p, q \in K\{Y\} \),

\[
\text{ord}(q) < \text{ord}(p) \implies (\exists x \in K) [p(x) = 0 \land q(x) \neq 0],
\]

where the order \( r = \text{ord}(p) \) is the largest derivative \( \delta^r Y \) used in \( p \).

This theory \( \text{DCF}_0 \) is complete and decidable and has quantifier elimination. Moreover, it has computable models:

**Theorem (Harrington, 1974)**

For every computable differential field \( k \), there exists a computable model \( K \) of \( \text{DCF}_0 \) and a computable embedding \( g \) of \( k \) into \( K \) such that \( K \) is a differential closure of the image \( g(k) \).
Noncomputable Differentially Closed Fields

By analogy to $\text{ACF}_0$, one may guess that all countable models of $\text{DCF}_0$ have computable presentations. However, it is known that there exist $2^\omega$-many (non-isomorphic) countable models of $\text{DCF}_0$. Indeed:

**Theorem (Marker-M.)**

For every countable graph $G$, there exists a countable $K \models \text{DCF}_0$ with

$$\text{Spec}(K) = \{ d : d' \text{ can enumerate the edges in some } G^* \cong G \}.$$
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It is not difficult to show that, for every $G$, there is another graph $H$ s.t.

$$\{d : d' \text{ enumerates the edges in some } G^* \cong G\} = \{d : d' \in \text{Spec}(H)\},$$

and that conversely, for each $H$, there is some such $G$. So the theorem proves that every countable graph $H$ yields a $K \models \mathbb{DCF}_0$ with

$$\text{Spec}(K) = \{d : d' \in \text{Spec}(H)\}.$$
Coding a Graph $G$ into $K \models DCF_0$

Start with a copy $\hat{Q}$ of the differential closure of $\mathbb{Q}$. Let $A$ be the following infinite set of indiscernibles in $\hat{Q}$:

$$A = \{a_0, a_1, \ldots\} = \{y \in \hat{Q} : \delta y = y^3 - y^2 \land y \neq 0 \land y \neq 1\}.$$

Each $a_m \in A$ will represent the node $m$ from $G$.

Let $E_{am an}$ be the elliptic curve defined by the equation

$$y^2 = x(x - 1)(x - a_m - a_n).$$

The coordinates of all solutions to this curve in $(\hat{Q})^2$ are algebraic over $\mathbb{Q}\langle a_m + a_n \rangle$ and $E_{am an}$ forms an abelian group, with exactly $j^2$ $j$-torsion points for every $j$, and with no non-torsion points. There is a homomorphism of differential algebraic groups from $E_{am an}$ into a vector group, whose kernel $E_{am an}^\dagger$ is called the Manin kernel of $E_{am an}$. 
Coding a Graph $G$ into $K \models DCF_0$

For each $m < n$ with an edge in $G$ from $m$ to $n$, add a generic point of $E_{a_m + a_n}^\#$ to our differential field. The coordinates of this point will each be transcendental over $\mathbb{Q}\langle a_m + a_n \rangle$. Let $K$ be the differential closure of the resulting differential field.

Thus the coding is:

$$G \text{ has an edge from } m \text{ to } n \iff (\exists (x, y) \in E_{a_m a_n}^\#) [x \text{ is transcendental over } \mathbb{Q}\langle a_m + a_n \rangle].$$

In particular, the points we added do not accidentally give rise to any transcendental solutions to any other $E_{a_m', a_n'}^\#$. 

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Spectra of $DCF_0$

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Spec\((K) = \{ d : d' \text{ enumerates some } G^* \cong G \}\)

Now if \(d\) is the degree of a copy \(K^* \cong K\), then with a \(d'\)-oracle, we enumerate the edges in some \(G^*\) as follows. Find all elements \(a_m^*\) of the set \(A^*\) of indiscernibles in \(K^*\), go through all solutions to \(E_{a_m^*a_n^*}\) for each \(m < n\), and ask whether each is transcendental over \(\mathbb{Q}\langle a_m^*, a_n^*\rangle\) and lies in \(E^\#_{a_m^*a_n^*}\). If we ever get an answer "YES," we enumerate \((m, n)\) into the edge relation of the graph \(G^*\). Thus \(G^* \cong G\): the isomorphism comes from restricting the isomorphism \(K^* \to K\) to \(A^* \to A\).

Conversely, if \(D \in d\) and \(D'\) enumerates the edges in some \(G^* \cong G\), we build \(K^* \cong K\) using a \(d\)-oracle. Start building \(\hat{\mathbb{Q}}^*\), finitely much at each step. At stage \(s\), if it appears (from \(D\)) that \(D'\) has enumerated an edge \((m, n)\) in \(G^*\), add a point \(x_{mn} \in E^\#_{a_m^*a_n^*}\) which is not (yet) algebraic over \(\mathbb{Q}\langle a_m, a_n\rangle\). If \(D'\) later changes and wipes out this enumeration, we can still make \(x_{mn}\) a \(t\)-torsion point for some large \(t\), hence algebraic. Finally, use Harrington’s theorem to build a \(D\)-computable differential closure \(K^*\) of the \(D\)-computable differential field defined here.
Low and Nonlow Degrees

For every $d' > 0'$, there exists a graph $G$ such that $d'$ enumerates a copy of $G$, but $0'$ does not. Therefore:

**Corollary**

For every nonlow degree $d$ (i.e., with $d' > 0'$), there exists some $K \models \text{DCF}_0$ of degree $d$ such that $K$ is not computably presentable.

We now prove the converse:

**Theorem (Marker-M.)**

Every low model of $\text{DCF}_0$ is isomorphic to a computable one.

This recalls the famous theorem of Downey-Jockusch that every low Boolean algebra is isomorphic to a computable one.
Principal Types over $k$

Over a field $E$, the principal 1-types are generated by the formulas $p(X) = 0$, where $p \in E[X]$ is irreducible. Over a differential field $k$, this is not enough! Over $\mathbb{Q}$, the differential polynomial $(\delta Y - Y)$ is irreducible, but only the following formula generates a principal type:

$$\delta Y - Y = 0 \ \& \ \ Y \neq 0.$$

In general, we need pairs $(p, q)$ from $k\{Y\}$, with $\text{ord}(p) > \text{ord}(q)$. If the formula $p(Y) = 0 \neq q(Y)$ generates a principal type, then $(p, q)$ is a constrained pair, and $p$ is constrainable. Every principal type is generated by a constrained pair, but not all irreducible $p(Y)$ are constrainable. $p(Y) = \delta Y$ is a simple counterexample.

**Fact**

$p \in k\{Y\}$ is constrainable $\iff$ $p$ is the minimal differential polynomial of some $x$ in the differential closure $K$ of $k$.

It is $\Pi^k_1$ for $(p, q)$ to be constrained, and $\Sigma^k_2$ for $p$ to be constrainable.
Low Differentially Closed Fields $K$

If $K$ is low, then the (computable infinitary) $\Pi^0_1$-theory of $K$ has degree $0'$, hence is computably approximable. This allows us to “guess” effectively at the minimal differential polynomial of any $x \in K$ over the differential subfield $\mathbb{Q}\langle x_{i_0}, \ldots, x_{i_n} \rangle \subseteq K$ generated by an arbitrary finite tuple from $K$.

Writing $K = \{x_0, x_1, \ldots\}$ and guessing thus, we build a computable differential field $F = \{y_0, y_1, \ldots\}$ and finite partial maps $h_s : K \rightarrow F$ such that:

- $(\forall n) \lim_s h_s(x_n)$ exists; and
- $(\forall m) \lim_s h_s^{-1}(y_m)$ exists; and
- $\forall s$ $h_s$ is a partial isomorphism, based on the approximations in $K$ to the minimal differential polynomials of its domain elements.

Thus $h = \lim_s h_s$ will be an isomorphism from $K$ onto $F$. 

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# Differences from Boolean Algebras

The Downey-Jockusch Theorem has been extended.

**Theorem (Downey-Jockusch; Thurber; Knight-Stob)**

Every low$_4$ Boolean algebra is isomorphic to a computable one.

In contrast, the first Marker-M theorem established that every nonlow Turing degree computes some $K \models \text{DCF}_0$ with $0 \notin \text{Spec}(K)$.

**Fact**

There exists a low Boolean algebra which is not $0'$-computably isomorphic to any computable Boolean algebra. (Downey-Jockusch always gives a $0''$-computable isomorphism.)

But the theorem for low differentially closed fields built a $\Delta_2$ isomorphism onto the computable copy.
Relativizing the Result

Relativizing the previous theorem yields:

**Corollary**

For every $K \models \text{DCF}_0$, $\text{Spec}(K)$ respects the equivalence relation $c \sim_1 d$ defined by $c' = d'$.

Proof: If $c \in \text{Spec}(K)$ and $d' = c'$, then $d$ can guess effectively at the minimal differential polynomials in the $c$-computable copy of $K$, and the process in the theorem builds a $d$-computable copy of $K$.

**Corollary (cf. Andrews, Montalbán, unpublished, using Richter)**

For every $K \models \text{DCF}_0$, $\text{Spec}(K)$ cannot be contained within any upper cone of Turing degrees, except the cone above $0$.

Proof: no other upper cone respects $\sim_1$.  

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Why Is This a Converse?

Corollary (Marker-M.)

For a set $S$ of Turing degrees, TFAE:

1. $S$ is the spectrum of some $K \models \text{DCF}_0$.
2. $S$ is the spectrum of some ANT graph and $S$ respects $\sim_1$.
3. $S$ is the preimage under jump of the spectrum of some ANT graph.

(ANT: automorphically non-trivial.)

$(1 \implies 2)$ was the relativized version of the second theorem (plus the HKSS theorem), and $(3 \implies 1)$ was the first theorem. For $(2 \implies 3)$, if $S = \text{Spec}(G)$, let $H$ be the jump of the structure $G$ (defined in work of Montalbán and Soskov-Soskova). By HKSS, we may take $H$ to be a graph. Then $\text{Spec}(H) = \{c' : c \in \text{Spec}(G)\}$, and so

$$\text{Spec}(G) \subseteq \{d : d' \in \text{Spec}(H)\}.$$ 

For $\supseteq$, if $d' \in \text{Spec}(H)$, then $d' = c'$ for some $c \in \text{Spec}(G)$, and $d \in \text{Spec}(G)$ since $\text{Spec}(G) = S$ respects $\sim_1$. 

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