The Cardinality of an Oracle in
Blum-Shub-Smale Computation

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(Joint work with Wesley Calvert, Murray State University,
and Ken Kramer, CUNY.)

Slides available at
qc.edu/~rmiller/slides.html
BSS Computation on $\mathbb{R}$

Roughly, a BSS machine $M$ on $\mathbb{R}$ operates like a Turing machine, but with a real number in each cell, rather than a bit.

- $M$ can compute full-precision $+$, $-$, $\cdot$, and $\div$ on numbers in its cells.
- $M$ can compare 0 to the number in any cell, using $=$ or $<$, and fork according to the answer.
- $M$ is allowed finitely many real numbers $z_0, \ldots, z_m$ as parameters in its program. The input and output (if $M$ halts) are tuples $\vec{y} \in \mathbb{R}^\infty = \{\text{finite tuples from } \mathbb{R}\}$.

A subset $S \subseteq \mathbb{R}^\infty$ is BSS-decidable iff its characteristic function $\chi_S$ is computable by a BSS machine, and BSS-semidecidable iff $S$ is the domain of some BSS-computable function.
Basic Facts about BSS Computation

For a machine $M$ with parameters $\vec{z}$, running on input $\vec{y}$, only elements of the field $\mathbb{Q}(\vec{z}, \vec{y})$ can ever appear in the cells of $M$.

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For each input $\vec{y}$, every $f_{i,s}(Y_1, \ldots, Y_n)$ is a rational function with coefficients from the field $\mathbb{Q}(\vec{z})$. If the input $\{y_1, \ldots, y_n\}$ is algebraically independent over $\mathbb{Q}(\vec{z})$, then each $f_{i,s}(\vec{Y})$ is uniquely defined.
Restrictions on BSS Computation

Given a machine $M$ with parameters $\vec{z}$, choose any input $\vec{y}$ algebraically independent over $\mathbb{Q}(\vec{z})$. If $M(\vec{y})$ halts after $t$ steps, then only finitely many functions $f_{i,s}$ appear. So there is an $\epsilon > 0$ such that for all inputs $\vec{x}$ within $\epsilon$ of $\vec{y}$, $M$ at stage $s$ contains:

$$
\begin{array}{cccccc}
  f_{0,s}(\vec{x}) & \cdots & f_{m,s}(\vec{x}) & f_{m+1,s}(\vec{x}) & \cdots & f_{m+n,s}(\vec{x}) & f_{m+n+1,s}(\vec{x}) & \cdots
\end{array}
$$

with the same functions $f_{i,s}$ as for $\vec{y}$.

Therefore, on an $\epsilon$-ball around $\vec{y}$ in $\mathbb{R}^n$, $M$ always halts after $t$ steps, and computes the function $\langle f_{0,t}(\vec{x}), \ldots, f_{m+n+t,t}(\vec{x}) \rangle$. 

Corollary: No BSS-decidable set can be dense and codense within any nonempty open subset of $\mathbb{R}^n$. 

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**Corollary:** No BSS-decidable set can be dense and codense within any nonempty open subset of $\mathbb{R}^n$. 
Oracle BSS-Machines

To do the same for a machine $M$ with parameters $\vec{z}$ and an oracle $C \subseteq \mathbb{R}^\infty$, we would have to ensure that $|\vec{x} - \vec{y}| < \epsilon$ and also, for all $s$,

$$(\forall i_0, \ldots, i_m) \left[ \langle f_{i_k, s}(\vec{x}) : k \leq m \rangle \in C \iff \langle f_{i_k, s}(\vec{y}) : k \leq m \rangle \in C \right].$$

Then the computation will fork exactly the same for $\vec{x}$ as for $\vec{y}$, and will output $\langle f_{i, t}(\vec{x}) \rangle$. 

**Theorem:** Let $H = \{ \langle \vec{p}; \vec{x} \rangle : \text{Program } \vec{p} \text{ halts on input } \vec{x} \}$ be the BSS Halting Problem. If $\chi_H$ is computable by a BSS program with oracle $C \subseteq \mathbb{R}^\infty$, then $|C| = 2^{\aleph_0}$.

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This answers a question from Meer and Ziegler.
Assume that the oracle \( C \subseteq \mathbb{R}^\infty \) has \(|C| < 2^{\aleph_0}\). For any oracle machine \( M \) with parameters \( \vec{z} \) and oracle \( C \), we claim that \( M^C \) does not compute \( \chi_H \).

Let \( p \) be the program which, on input \( \langle a, b \rangle \), halts iff \( b \) is algebraic over \( \mathbb{Q}(a) \). Fix any \( y_0, y_1 \in \mathbb{R} \) algebraically independent over the field \( E \) (of size \( < 2^{\aleph_0} \)) generated by \( \vec{z} \) and \( p \) and all tuples in \( C \). Let \( R \) be the finite set of rational functions \( f \in E(Y_0, Y_1) \) such that \( f(y_0, y_1) \) appears in a cell during this computation. Fix \( n \in \mathbb{N} \) such that each \( f \in R \) is a quotient of polynomials of degree \( < n \).
Proving the Theorem

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Now \( \langle p, y_0, y_1 \rangle \notin \mathbb{H} \), by algebraic independence, so \( M^C(p, y_0, y_1) = 0 \). We want to choose \( \langle p, x_0, x_1 \rangle \in \mathbb{H} \) close to \( \langle p, y_0, y_1 \rangle \) to fool \( M^C \) into computing \( M^C(p, x_0, x_1) = 0 \) as well.
Proving the Theorem

Recall: \( y_0, y_1 \in \mathbb{R} \) independent over \( E \); finite set \( R \subset E(Y_0, Y_1) \); all \( f \in R \) have \( f = \frac{g}{h} \) of degree \(< n\).

Now choose \( x_0 \) transcendental over \( E \), and \( x_1 = m\sqrt{x_0} + q \), with \( m > n \) prime and \( q \in \mathbb{Q} \) so that \( x_0, x_1 \) are sufficiently close to \( y_0, y_1 \). So \( x_1 \) has degree \( m \) over \( E(x_0) \). Now for \( f = \frac{q}{h} \in R \),

\[
f(\vec{x}) = c \in E \implies g(\vec{x}) - ch(\vec{x}) = 0 \implies (g - ch) = 0 \text{ in } E[Y_0, Y_1].
\]

So \( f = \frac{q}{h} = c \) is constant. Thus

\[
f(x_0, x_1) \in E \iff f \text{ is constant} \iff f(y_0, y_1) \in E.
\]

So the computation by \( M^C \) on input \( \langle p, x_0, x_1 \rangle \) follows the same path as on \( \langle p, y_0, y_1 \rangle \), and outputs the same answer: \( \langle p, x_0, x_1 \rangle \notin \mathbb{H} \). This is wrong!
When can a countable set decide an uncountable (and co-uncountable) set?

Easy answer: \( \{ x \in \mathbb{R} : x > 0 \} \) is BSS-decidable. (Is there a similar subset of \( \mathbb{C} \), for BSS-computation on \( \mathbb{C} \)?) Indeed, \( \{ x \in \mathbb{R} : x \in (0,1] \ \& \ x \text{ begins with an even number of 0's} \} \) is BSS-decidable. This is the set \[ \left[ \frac{1}{32}, \frac{1}{16} \right] \cup \left[ \frac{1}{8}, \frac{1}{4} \right] \cup \left[ \frac{1}{2}, 1 \right]. \]
Shall We Generalize?

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Local Bicardinality

Defn.: A set $S \subseteq \mathbb{R}$ is *locally of bicardinality* $\leq \kappa$ if there exist two open
subsets $U$ and $V$ of $\mathbb{R}$ with $|\mathbb{R} - (U \cup V)| \leq \kappa$ and $|U \cap S| \leq \kappa$ and $|V \cap \overline{S}| \leq \kappa$.
The *local bicardinality* of $S$ is the least cardinal $\kappa$ such that $S$ is locally
of bicardinality $\leq \kappa$.

So both $S$ and $\overline{S}$ are open, up to a set of size $\kappa$. Notice that the open
set $(U \cap V)$ is empty, since

$$|U \cap V| \leq |U \cap S| + |V \cap \overline{S}| \leq \kappa.$$  

(Question: is there an equivalent but simpler definition?)

**Example:** The Cantor middle-thirds set has local bicardinality $2^{\aleph_0}$. 
**Thm.:** If $C \subseteq \mathbb{R}^\infty$ is an oracle set of infinite cardinality $\kappa < 2^{\aleph_0}$, and $S \subseteq \mathbb{R}$ is a set with $S \leq_{\text{BSS}} C$, then $S$ must be locally of bicardinality $\leq \kappa$. The same holds for oracles $C$ of infinite co-cardinality $\kappa < 2^{\aleph_0}$.

Proof: Consider $\chi_S(y) = M^C(y)$ for any $y$ transcendental over the subfield $E$ generated by $C$. On some open interval $B(y)$, $\chi_S(x) = \chi_S(y)$ for every $x \in B(y)$ transcendental over $E$, so either $|S \cap B(y)| \leq \kappa$ or $|\bar{S} \cap B(y)| \leq \kappa$. Also, if $B(y) \cap B(y') \neq \emptyset$, then $\chi_S(y) = \chi_S(y')$. So let

$$U = \bigcup \{B(y) : y \notin S\} \quad V = \bigcup \{B(y) : y \in S\}.$$  

So $|U \cup V| \leq |E| = \kappa$. If we assume all $B(y)$ to have rational end points, then these are both countable unions, and hence $(U \cap S)$ is a countable union of sets $(B(y) \cap S)$ of size $\leq \kappa$; likewise for $(V \cap \bar{S})$. 

Complex Numbers

A BSS-machine on $\mathbb{C}$ can perform the field operations, but there is no instruction for deciding whether “$z > 0$.” Here the theorem is nicer (and easily proven):

**Thm.** : If $C \subseteq \mathbb{C}^\infty$ is an oracle set of infinite cardinality $\kappa$, and $S \subseteq \mathbb{C}$ with $S \leq_{BSS} C$, then either $|S| \leq \kappa$ or $|\overline{S}| \leq \kappa$. In particular, for all $x, y$ transcendental over $C$, we have

$$x \in S \iff y \in S.$$  

This fails for sets $S \subseteq \mathbb{C}^2$: just consider the BSS-decidable set

$$\{\langle z, z \rangle : z \in \mathbb{C}\}.$$  

Similarly for subsets of $\mathbb{R}^2$, the theorem on local bicardinality fails. We believe that this can be fixed by considering size-$\kappa$ unions of Zariski-closed subsets of $\mathbb{C}^2$ and $\mathbb{R}^2$, and generally for $\mathbb{C}^\infty$ and $\mathbb{R}^\infty$. 
**Other Results**

**Thm.:** Let

$$A_d = \{y \in \mathbb{R} : y \text{ is algebraic of degree } d \text{ over } \mathbb{Q}\}.$$ 

Then for all $d \geq 0$, $A_{d+1} \not\leq_{BSS} A_d$. Indeed $A_{d+1} \not\leq_{BSS} \bigcup_{c \leq d} A_c$. 

**Prop.:** Let $p$ and $r$ be any positive integers. Then $A_p \leq_{BSS} A_r$ if and only if $p$ divides $r$. 

**Prop.:** Let $P$ be the set of all prime numbers in $\omega$ and let $S \subseteq P$ and $T \subseteq P$. Then $S \subseteq T$ if and only if $A_S \leq_{BSS} A_T$. 

(Here $A_S = \bigcup_{d \in S} A_d$.)

**Cor.:** There exists a subset $L$ of the BSS-semidecidable degrees such that $(L, \leq_{BSS}) \sim (P(\omega), \subseteq)$. 

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Cardinality of an Oracle  
CCA 2010  
12 / 13
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  Then for all \( d \geq 0 \), \( \mathbb{A}_d \nsubseteq BSS \mathbb{A}_d \). Indeed \( \mathbb{A}_{d+1} \nsubseteq BSS \cup_{c \leq d} \mathbb{A}_c \).
- **Prop.:** Let \( p \) and \( r \) be any positive integers. Then \( \mathbb{A}_p \leq BSS \mathbb{A}_r \) if and only if \( p \) divides \( r \).
- **Prop.:** Let \( P \) be the set of all prime numbers in \( \omega \) and let \( S \subseteq P \) and \( T \subseteq P \), Then \( A_S \leq BSS A_T \) if and only if \( S \subseteq T \).
  (Here \( \mathbb{A}_S = \bigcup_{d \in S} \mathbb{A}_d \).)
- **Cor.:** There exists a subset \( \mathcal{L} \) of the BSS-semidecidable degrees such that \( (\mathcal{L}, \leq_{BSS}) \cong (\mathcal{P}(\omega), \subseteq) \).
Online Help

- Introduction to BSS computation:
  L. Blum, F. Cucker, M. Shub, and S. Smale; *Complexity and Real Computation* (Berlin: Springer-Verlag, 1997).

- Relevant papers:

- Full version of these results, joint with Calvert & Kramer, available at [qc.edu/~rmiller/BSSfull.pdf](http://qc.edu/~rmiller/BSSfull.pdf)

- These slides available at [qc.edu/~rmiller/slides.html](http://qc.edu/~rmiller/slides.html)