Local Computability and Uncountable Structures

Russell Miller,
Queens College &
Graduate Center – CUNY

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Local Descriptions of Structures

Defn.: A simple cover $\mathcal{A}$ of a structure $\mathcal{S}$ is a set $
\{\mathcal{A}_i : i \in I\}$ which contains the finitely generated substructures of $\mathcal{S}$, up to isomorphism.

$\mathcal{A}$ is computable if every $\mathcal{A} \in \mathcal{A}$ is.

$\mathcal{A}$ is uniformly computable if there is a single algorithm listing out all $\mathcal{A}_i$ in $\mathcal{A}$. In this case $\mathcal{S}$ is locally computable.

Examples:

- All fields, and all relational structures, have computable simple covers.

- The ordered field $(\mathbb{R}, <)$ does not.

- The ordered field of computable real numbers is not locally computable, but has a computable simple cover.
Let $\mathcal{S}$ be locally computable via $\{\mathcal{A}_0, \mathcal{A}_1, \ldots\}$. Suppose $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{S}$ are finitely generated. If

\[ \begin{array}{ccc}
\mathcal{B} & \subseteq & \mathcal{C} \\
\beta \cong & & \gamma \cong \\
\mathcal{A}_i & \mapsto & \mathcal{A}_j
\end{array} \]

commutes, we say that $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ lifts to the inclusion $\mathcal{B} \subseteq \mathcal{C}$ via the isomorphisms $\beta$ and $\gamma$.

**Defn.** A cover of $\mathcal{S}$ also has sets $I_{ij}^\mathcal{A}$ of embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_j$, such that every inclusion in $\mathcal{S}$ is the lift of some $f$ in some $I_{ij}^\mathcal{A}$, and every $f \in I_{ij}^\mathcal{A}$ lifts to an inclusion in $\mathcal{S}$.

The cover is *uniformly computable* if all $I_{ij}^\mathcal{A}$ are c.e. uniformly in $i$ and $j$.

Notice that $f$ is determined by its values on the generators of $\mathcal{A}_i$. 
Examples

- Every infinite linear order has the same uniformly computable cover: $\mathcal{A}_i$ is the linear order on $i$ elements, and $I_{ij}$ contains all embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_j$.

- In $\mathbb{C}$, the cover contains every f.g. field of characteristic 0, and every possible embedding $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ lifts to an inclusion. Similarly for any ACF, given its transcendence degree.

- $\mathbb{R}$ also has a uniformly computable cover. This follows from:

  **Lemma:** $S$ has a uniformly computable cover iff $S$ has a uniformly computable simple cover.

  **Proof:** Given a simple cover $\{\mathcal{A}_i\}$, consider the cover containing all f.g. substructures of each $\mathcal{A}_i$, with inclusion maps from these substructures into the original $\mathcal{A}_i$. 

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1-Extensionality

**Defn.**: Every embedding from any $A_i$ into $S$ is 0-extensional. An isomorphism $\beta : A_i \hookrightarrow B \subseteq S$ is 1-extensional if

- $(\forall j)(\forall f \in I^2_{i,j})(\exists C \subseteq S)[f \text{ lifts to } B \subseteq C \text{ via } \beta \text{ and some isomorphism } \gamma]$; and
- $(\forall \text{ f.g. } C \supseteq B)(\exists j)(\exists f \in I^2_{i,j})[f \text{ lifts to } B \subseteq C \text{ via } \beta \text{ and some isomorphism } \gamma]$.

Intuition: A 1-extensional $\beta$ is a strong pairing between $A_i$ and $B$, in that $\mathcal{A}$’s ways to extend $A_i$ are exactly the ways of extending $B$ within $S$.

$\mathcal{A}$ is a 1-extensional cover if every $A_i \in \mathcal{A}$ is the domain of a 1-extensional embedding and every f.g. $B \subseteq S$ is the range of one.
Example

**Cantor Space:** The linear order on \(2^\omega\) has a 1-extensional cover. The objects are all finite linear orders \(a_0 \prec \cdots \prec a_n\) under the following specifications. \(a_0\) may or may not be designated as the left end point; likewise \(a_n\) as the right end point. Each \(a_m\) not so designated may be called either a left gap point or a right gap point (but not both). If \(a_m\) is a LGP and \(a_{m+1}\) a RGP, then we must specify whether they belong to the same gap or not.

An embedding \(f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j\) belongs to \(I_{i,j}^{\mathcal{A}}\) if it respects all these properties: \(a_m\) is a left end point iff \(f(a_m)\) is, etc.

So, if \(a_m\) and \(a_{m+1}\) are LGP and RGP for the same gap, then there can be no element between \(f(a_m)\) and \(f(a_{m+1})\) in \(\mathcal{A}_j\).
\textbf{θ-Extensionality}

**Defn.**: Let θ be an ordinal. An isomorphism 
\( \beta : A_i \hookrightarrow B \subseteq S \) is θ-extensional if

- \((\forall \text{ f.g. } C \supseteq B)(\forall \zeta < \theta)(\exists j)(\exists f \in I_{i,j}^A) \]
  \[f \text{ lifts to } B \subseteq C \text{ via } \beta \text{ and a } \zeta\text{-extensional } \gamma].

- and \((\forall j)(\forall f \in I_{i,j}^A)(\forall \zeta < \theta)(\exists C \subseteq S) \]
  \[f \text{ lifts to } B \subseteq C \text{ via } \beta \text{ and a } \zeta\text{-extensional } \gamma];

Intuition: A θ-extensional \( \beta \) is a strong pairing
between \( A_i \) and \( B \), in that \( \mathcal{A} \)'s ways to extend \( A_i \)
are exactly the ways of extending \( B \) within \( S \)
while preserving the \( \Sigma_\zeta \)-theory over \( B \).

\( \mathcal{A} \) is a \textit{θ-extensional cover} if every \( A_i \in \mathcal{A} \) is the
domain of an θ-extensional embedding and every
f.g. \( B \subseteq S \) is the range of one.
Lemma: $\mathbb{R}$ has no 1-extensional cover.
Proof: If $\mathcal{A}$ were such a cover, fix a noncomputable $x \in \mathbb{R}$ and a 1-extensional $\beta : A_i \hookrightarrow Q(x) \subseteq \mathbb{R}$. Then for $q \in \mathbb{Q}$:

\[ q < x \iff \exists y \in \mathbb{R} \; y^2 = x - q \]

\[ \iff \exists j \; \exists f \in I_{ij}^x \; \exists a \in A_j \]
\[ [a^2 = f(\beta^{-1}(x)) - f(\beta^{-1}(q))] \]

So the lower cut defined by $x$ would be computably enumerable, and similarly for the upper cut.
Theorem (Miller): Suppose $S$ has a $\theta$-extensional cover. Then $(\forall \zeta \leq \theta)$, and for any finite set $\vec{p}$ of parameters in $S$, the $\Sigma_\zeta$-theory of $(S, \vec{p})$ is arithmetically $\Sigma^0_\zeta$, uniformly in $i$ and $\alpha^{-1}(\vec{p})$, where $\alpha : A_i \hookrightarrow \langle \vec{p} \rangle$ is $\theta$-extensional.

Moreover, this applies even to infinitary computable $\Sigma_\zeta$ formulas over $P$. 

$\Sigma_\theta$-Theory of $S$
Now we want to be able to extend our diagrams infinitely far to the right.

**Defn.:** A set $M$ of embeddings $\beta : A_i \hookrightarrow S$ is a **correspondence system** if:

1. $(\forall i)(\exists \beta \in M) A_i = \text{dom}(\beta)$; and
2. $(\forall f.g. B \subseteq S)(\exists \beta \in M) B = \text{range}(\beta)$; and

and for all maps $\beta : A_i \cong B$ in $M$:

1. $(\forall j \forall f \in I_{i,j}^{\text{cf}})(\exists C \supseteq B)[f \text{ lifts to the inclusion } B \subseteq C \text{ via } \beta \text{ and some } \gamma \in M]$; and
2. $(\forall f.g. C \supseteq B)(\exists j \exists f \in I_{i,j}^{\text{cf}})[f \text{ lifts to the inclusion } B \subseteq C \text{ via } \beta \text{ and some } \gamma \in M]$.

**Defn.:** A structure is **$\infty$-extensionally locally computable** if it has a correspondence system over a uniformly computable cover.
Perfect Local Computability

$M$ is perfect if, for all $\beta, \gamma \in M$ with $\text{range}(\beta) = \text{range}(\gamma)$, we have $(\gamma^{-1} \circ \beta) \in I^{\kappa}_{ij}$, where $A_i = \text{dom}(\beta)$ and $A_j = \text{dom}(\gamma)$.

- The uniformly computable cover we built for $C$ has a perfect correspondence system.
- The uniformly computable cover we built for Cantor space (as a linear order) is perfect.
- It is also possible to view Cantor space as the top level of the tree $2^{<\omega+1}$, as a partial order, and to build a perfect correspondence system for this structure.

Such structures are called perfectly locally computable.
Globally Computable Structures

**Theorem** (Miller): For a countable structure $S$, TFAE:

1. $S$ is computably presentable;
2. $S$ is perfectly locally computable;
3. $S$ has a uniformly computable cover with a correspondence system, satisfying AP.

Proof: For (1 $\implies$ 2), build the *natural cover* $\mathcal{A}$ containing all f.g. substructures of $S$, under inclusion.

For (2 $\implies$ 3), all perfect covers have AP.

For (3 $\implies$ 1), amalgamate the $\mathcal{A}_i$ together over all embeddings in $\mathcal{A}$, to get a computable presentation of $S$. 
\(\infty\)-Extensionality

(joint work with Dustin Mulcahey)

**Lemma:** Let structures \(\mathcal{C}\) and \(\mathcal{S}\) have correspondence systems over the same cover. Suppose that \(\mathcal{C}\) is countable, and that \(P\) is a countable subset of \(\mathcal{S}\). Then there exists an elementary embedding of \(\mathcal{C}\) into \(\mathcal{S}\) whose image contains \(P\).

**Corollary:** Any two countable structures with correspondence systems over the same cover are isomorphic.
Simulations

**Defn.:** A *simulation* $\mathcal{C}$ of a structure $\mathcal{S}$ is an elementary substructure of $\mathcal{S}$ which realizes the same $n$-types as $\mathcal{S}$ (for all $n$).

If for every $\vec{a} \in \mathcal{C}$ there is $\vec{p} \in \mathcal{S}$ such that $\mathcal{C}$ and $\mathcal{S}$ realize the same $n$-types over $\vec{a}$ and $\vec{p}$, and likewise for every $\vec{p}$ there is an $\vec{a}$, then $\mathcal{C}$ simulates $\mathcal{S}$ *over parameters*.

**Examples:** The algebraic closure of the field $\mathbb{Q}(X_0, X_1, \ldots)$ is a computably presentable simulation of $\mathbb{C}$ over parameters.

The intersection of $\mathbb{Q}$ with Cantor space ($\subset [0, 1]$, as linear order) is a computably presentable simulation of Cantor space over parameters.
**Building Simulations**

**Lemma:** Every $\infty$-extensionally locally computable structure $S$ has a countable simulation $C$ over parameters with a correspondence system over the cover of $S$.

Proof: For each $i$, enumerate *one* image $\alpha(A_i)$ into $C$, with $\alpha$ in the correspondence system $M$ for $S$. Then close $C$ under the $\forall\exists$ conditions for a correspondence system.

Notice that if $M$ is perfect for $S$, then the new system is perfect for $C$. 
**Computable Simulations**

**Thm.** (Mulcahey-Miller): Every perfectly locally computable structure $\mathcal{S}$ has a computably presentable simulation $\mathcal{C}$ over parameters.

Moreover, if we fix a computable $\mathcal{D} \cong \mathcal{C}$, then for any countable parameter set $P \subseteq \mathcal{S}$, there exists an embedding $f_P : \mathcal{D} \hookrightarrow \mathcal{S}$ such that $P \subseteq \text{range}(f_P)$ and $\mathcal{S}$ and $f_P(\mathcal{D})$ realize exactly the same finitary types over every finite subset of the image of $f_P$. (We call $f_P$ an *elementary embedding over parameters*.)
Computable Simulations

**Thm.** A structure $S$ has an $\infty$-extensional cover with $AP \iff S$ has a computable simulation $C$ over parameters, such that, for all elementary embeddings $f : C \hookrightarrow S$ over parameters, all $\vec{a} \in C$, and all $x \in S$, there exists an elementary embedding $g : C \hookrightarrow S$ over parameters with $g|\vec{a} = f|\vec{a}$ and $x \in \text{range}(g)$.

The cover $A$ is the natural cover of $C$. The correspondence system contains all restrictions (to elements of $A$) of elementary embeddings of $C$ into $S$ over parameters.
**C and its Simulations**

A computable simulation of the field $\mathbb{C}$ must have infinite transcendence degree and be algebraically closed. Hence it must be the field $F = \mathbb{Q}(X_0, X_1, \ldots)$. However,

**Fact:** The natural cover of $F$ is *not* a perfect cover of $\mathbb{C}$. This follows from:

**Lemma:** A perfect cover of $\mathbb{C}$ must include a set $I_{ij}^\mathfrak{a}$ of size $> 1$.

Still, the natural cover $\mathfrak{A}$ of $F$ is an $\infty$-extensional cover of $\mathbb{C}$, and has AP. The correspondence system consists of all embeddings of every $A_i \in \mathfrak{A}$ into $\mathbb{C}$. 

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Cardinalities

Fix any countable sequence $\kappa_0 < \kappa_1 < \cdots$ of cardinals. Let $T$ be the tree of height $\omega$ with each node at level $n$ having $\kappa_n$-many immediate successors.

This $T$ is perfectly locally computable: $\mathcal{A}$ contains all finite substructures of $\omega^{<\omega}$, under embeddings which preserve levels, and $M$ contains all level-preserving embeddings $\mathcal{A}_i \hookrightarrow T$.

But we can make the $\kappa$-sequence arbitrarily complex!
Local Constructivizability

**Defn.** (Ershov): A structure $S$ is *locally constructivizable* if, for all finite tuples $\vec{p} \in S$, the $\exists$-theory of $(S, \vec{p})$ is arithmetically $\Sigma_1^0$.

**Cor.:** Every 1-extensional structure is locally constructivizable.

Local constructivizability may be seen as a non-uniform version of 1-extensional local computability.

The field $\mathbb{R}$ is locally computable, but not locally constructivizable.

The field of computable real numbers is locally constructivizable, and locally computable, but not 1-extensional. (The *ordered* field of computable real numbers is not even locally computable.)
Questions

1. Can there exist a structure $S$ with a computable simulation (over parameters?) such that $S$ is not perfectly locally computable? Or such that $S$ is not $\infty$-extensional with AP?

2. Develop a reasonable theory of maps (and computable maps) among covers.
   - Functors?

3. How locally computable is the structure $(\mathbb{C}, +, \cdot, 0, 1, f)$, where $f(z) = e^z$? (Similar questions for other holomorphic functions.)

4. Find $\theta$-extensionally locally computable structures which are not $(\theta + 1)$-extensional, and which have arbitrarily complex $\Sigma_{\theta+1}$-theory over parameters.