Turing Degree Spectra of Real Closed Fields

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(Joint work with Victor Ocasio Gonzalez, UPR-Mayaguez.)
Spectra of Countable Structures

Let $S$ be a structure with domain $\omega$, in a finite language.

**Definition**

The *Turing degree* of $S$ is the join of the Turing degrees of the functions and relations on $S$. If these are all computable, then $S$ is a *computable structure*.

**Definition**

The *spectrum* of $S$ is the set of all Turing degrees of copies of $S$:

$\text{Spec}(S) = \{ \text{deg}(M) : M \cong S \& \text{dom}(M) = \omega \}$.

So the spectrum measures the level of complexity intrinsic to the structure $S$. 
Spectra for Different Classes

- Every spectrum of an automorphically non-trivial structure, in a computable language, is the spectrum of a graph, a lattice, a group, a partial order, and a field. (Results by HKSS and MPSS.)
- In particular, every upper cone of degrees, \( \{ \text{all high}_n \text{ degrees} \} \), \( \{ \text{all non-low}_n \text{ degrees} \} \), \( \{ \text{all nonzero degrees} \} \), \( \{ \text{all non-hyperarithmetic degrees} \} \) are spectra of graphs.
- A Boolean algebra cannot have a low_4 degree in its spectrum unless it also has 0. (Downey-Jockusch, Thurber, Knight-Stob.)
- BA’s, trees, and linear orders cannot realize an upper cone as a spectrum (Richter). However, LO’s can have a spectrum containing any given \( d > 0 \) and not containing 0.
- The spectrum of an ACF always contains all degrees.
- The spectra of models of \( \text{DCF}_0 \) are precisely the preimages under jump of the spectra of graphs. (Marker-M.)
- Spectra of algebraic fields and rank-1 torsion-free abelian groups are defined by the ability to enumerate some specific subset of \( \omega \).
Real Closed Fields

**Definition**

A *real closed field* $F$ is a model of the theory of the real numbers $(\mathbb{R}, 0, 1, +, \cdot)$. The *positive* field elements are those nonzero elements with square roots: this defines an order on $F$. The *finite* elements are those $x$ for which some natural number $n$ satisfies $-n < x < n$.

$F$ is *archimedean* if every $x \in F$ is finite. If not, then $F$ has both infinite and *infinitesimal* elements.

Every finite $x \in F$ defines a Dedekind cut in $\mathbb{Q}$, with left side $\{ q \in \mathbb{Q} : q < x \}$ and right side $\{ q \in \mathbb{Q} : x \leq q \}$.

The *residue field* $F_0$ of (a nonarchimedean) $F$ consists of one element realizing each Dedekind cut realized in $F$. If $F_0$ is just the real closure of $\mathbb{Q}$, then it is canonically a subfield of $F$. However, if $F_0$ contains transcendentals, then it has no canonical embedding into $F$. 
Computability and real closures

**Theorem (Ershov; Madison)**

For every $d$-computable ordered field $F$, there is a $d$-computable presentation of the real closure of $F$.

So, to give a $d$-computable presentation of the real closure of $F$, it suffices to present $F$ itself using a $d$-oracle.
Dedekind cuts

In any computable RCF, we can give a computable enumeration \( \langle A_{n,s}, B_{n,s} \rangle_{n,s \in \omega} \) of all Dedekind cuts \((A_n, B_n)\) realized in \(F\). We think of each cut as a decreasing sequence of intervals \((a_{n,s}, b_{n,s}]\), with \(a_{n,s} = \max(A_{n,s})\) and \(b_{n,s} = \min(B_{n,s})\). It is not difficult to make this enumeration injective.

**Theorem**

For an archimedean RCF \(F\), the following are equivalent:

- \(d \in \text{Spec}(F)\).
- \(d\) enumerates the Dedekind cuts realized in \(F\) as \((A_n, B_n)\), in such a way that the dependence relation on the realizations of these cuts is \(\Sigma^d_1\).
Upper Cones as Spectra

**Proposition (folklore)**
Every upper cone \( \{d : c \leq d\} \) of Turing degrees is the spectrum of a RCF.

Proof: given \( c \), find a real number \( x \) (necessarily transcendental, when \( c \neq 0 \)) whose Dedekind cut in \( \mathbb{Q} \) has degree \( c \). The real closure of \( \mathbb{Q}(x) \) is then \( c \)-presentable, but conversely, each of its presentations must compute the Dedekind cut of (the image of) \( x \), hence computes \( c \).

This distinguishes RCF’s from linear orders, trees, Boolean algebras, algebraic fields, and models of \( \text{ACF} \) and \( \text{DCF}_0 \), in terms of the spectra they can realize.
High degrees

Question: which families of Turing degrees are defined by the property of being able to realize a specific collection of Dedekind cuts?

**Theorem (Jockusch, 1972)**

The degrees $d$ which can enumerate the computable sets are precisely the high degrees (i.e., those with $d' \geq 0''$).
High degrees

Question: which families of Turing degrees are defined by the property of being able to realize a specific collection of Dedekind cuts?

Theorem (Jockusch, 1972)
The degrees $d$ which can enumerate the computable sets are precisely the high degrees (i.e., those with $d' \geq 0''$).

Theorem (Korovina-Kudinov)
The spectrum of the field of all computable real numbers contains precisely the high degrees.

This relativizes: the spectrum of the field of $c$-computable real numbers contains precisely those degrees $d$ with $d' \geq c''$. 
Proof: $\text{Spec}(\mathbb{R}_c) = \{\text{high degrees}\}$

$\Rightarrow$: If $d$ computes a copy of the field $\mathbb{R}_c$ of computable real numbers, then $d$ can list out all the Dedekind cuts realized in $\mathbb{R}_c$. From this list, one quickly gets an enumeration of all computable sets. So, by Jockusch’s result, $d$ is high.

$\Leftarrow$: If $d$ is high, then some $d$-computable function can approximate $0''$. We use this to guess, $d$-computably, whether each pair $(W_i, W_j)$ of c.e. subsets of $\mathbb{Q}$ constitutes a Dedekind cut or not. When it appears to be a cut (and when this cut becomes distinct from all previous cuts), we start building an element $x_{ij}$ in our presentation of $\mathbb{R}_c$ to realize that cut. If the approximation changes its mind, we can always turn $x_{ij}$ into a nearby rational element of our presentation, consistently with the finitely many facts so far defined about this presentation.
Dedekind cuts are not enough

**Theorem**

There exists an archimedean real closed field $F$ with a computable enumeration of all Dedekind cuts realized in $F$, yet with $\text{Spec}(F)$ containing precisely the high degrees.

The set $\text{Inf}$ is coded into $F$ in such a way that with any presentation of $F$ and with a transcendence basis for that presentation, one can decide $\text{Inf}$. We uniformly enumerate Dedekind cuts

\[ \{ (a_e, s, b_e, s) : e \in \omega \} \]

such that, for each $e$, $a_e = \lim_s a_{e,s}$ is transcendental over $\mathbb{Q}$ iff $W_e$ is infinite). In fact, if $W_e$ is infinite, then $a_e$ will be transcendental over the subfield $\mathbb{Q}(a_0, \ldots, a_{e-1})$.

Given any presentation of $F$, of degree $d$, a $d'$-oracle allows us to find an element realizing the cut $(a_{e,s}, b_{e,s})$, and to check transcendence of this element (which is $d'$-decidable).
Nonarchimedean real closed fields

In a nonarchimedean RCF, we partition the positive infinite elements into *multiplicative classes*:

\[ x \sim y \iff \exists n \left[ x < y^n \& y < x^n \right]. \]

These classes are linearly ordered in \( F \). Write \( L_F \) for this derived linear order, which is then presentable from the jump of each copy of \( F \).

An RCF \( F \) is *principal* if it is the smallest RCF with a given residue field \( F_0 \) and with a given linear order \( L \) as \( L_F \).
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**Theorem (Ocasio, Ph.D. thesis)**

For every \( L \), the principal RCF \( F \) with residue field \( RC(\mathbb{Q}) \) and derived linear order \( L \) satisfies

\[ \text{Spec}(F) = \{ d : d' \in \text{Spec}(L) \}. \]
A distinction on derived orders

Proposition

Suppose that the derived linear order $L_F$ of an RCF $F$ has a left end point. Then the property of being finite in $F$ is relatively intrinsically computable. (Hence so is being infinitesimal.)

Proof: Fix an element $y_0$ in the least positive infinite multiplicative class. Then $x$ is finite in $F$ iff $(\exists n)[−n < x < n]$; while $x$ is infinite in $F$ iff $(\exists m > 0) \ y_0 < x^m$.

Corollary

If $L_F$ has a left end point, then $\text{Spec}(F) \subseteq \text{Spec}(F_0)$.

Proof: $F_0$ is defined as the quotient of the ring of finite elements of $F$, modulo the ideal of infinitesimals in $F$. 
Spectra when $L_F$ has a left end point

Ocasio’s theorem shows that the containment in the Proposition does not reverse: we can have $\text{Spec}(F) \neq \text{Spec}(F_0)$.

**Theorem**

For every $L$ with a left end point, and every archimedean RCF $F_0$, the principal RCF $F$ with residue field $F_0$ and derived linear order $L$ satisfies

$$\text{Spec}(F) = \text{Spec}(F_0) \cap \{d : d' \in \text{Spec}(L)\}.$$  

The proof is essentially just Ocasio’s construction, with $\text{RC}(\mathbb{Q})$ replaced by $F_0$. 

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Spectra when $L_F$ has no left end point

**Theorem**

There exists a computable principal RCF $F$ whose residue field $F_0$ has no computable presentation. (By the previous theorem, $L_F$ has no left end point.)

The construction of $F$ builds a sequence of elements $y_e$, with

$$e \in \text{Fin} \iff (\exists q \in \mathbb{Q})((y_e - q) \text{ is infinitesimal}).$$

Use the complete binary tree $T$, guessing at level $e$ whether $e \in \text{Inf}$:

At each node $\alpha$ we have a $y_\alpha \in F$, which remains fixed from stage to stage. The set $\{y_\alpha : \alpha \in T\}$ is algebraically independent in $F$.

$y_e$ will equal $y_\alpha$ for that $\alpha$ on the true path at level $e$.

At each stage $s$, $y_\alpha$ is close to some $q_{\alpha,s} \in \mathbb{Q}$, with $x_{\alpha,s} = q_{\alpha,s} - y_\alpha$ positive and potentially infinitesimal.
0′ Construction

Whenever $W_{e,s+1}$ adds an element, we make the difference $x_{α,s}$ noninfinitesimal, so $y_α$ is not that close to $q_{α,s}$, and choose a new $q_{α,s+1} < q_{α,s}$ for $y_α$ to approximate. Making $x_{α,s}$ noninfinitesimal makes all $x_{β,s} > x_{α,s}$ noninfinitesimal as well, injuring those $β$.

So we choose $x_{α,s} < x_{β,s}$ iff $α < β$ on $T$:

\[
x_{∞∞∞} < x_{∞} < x_{∞f} < x_{λ} < x_{f∞} < x_{f} < x_{ff}
\]

\[ WV_2 \quad fin \quad fin \quad α \]
\[ W_1 \quad fin \quad fin \quad λ \]
\[ W_0 \quad fin \quad fin \]
Given a \(d\)-computable copy \(E_0\) of \(F_0\), a \(d'\)-oracle allows us to find the unique element \(z_0 \in E_0\) realizing the same cut as \(y_0 = y_\lambda\). Since \(\mathbb{Q}\) is \(d\)-c.e. inside \(E_0\), \(d'\) then tells us whether this \(z_0\) is rational in \(E_0\). If so, then \(0 \in \text{Fin}\); if not, then \(0 \in \text{Inf}\).

With this info, we know which \(\alpha_1\) at level 1 lies on the true path. Set \(y_1 = y_{\alpha_1}\), and find the unique \(z_1 \in E_0\) realizing the same cut as \(y_1\). If \(z_1 \in \mathbb{Q}\), then \(1 \in \text{Fin}\); else \(1 \in \text{Inf}\).

Continuing recursively, we compute \(\text{Inf}\) from the \(d'\)-oracle.
Conclusions and Questions

It remains open whether RCF’s can realize all possible spectra of automorphically nontrivial structures. This seems unlikely, but no counterexample is known.

There appears to be a tight connection between spectra of RCF’s and highness properties: such spectra are often defined by the ability of the jump $d'$ to compute some particular degree $c$. Can this be made explicit somehow?

Problem: does the spectrum of an RCF $F$ depend only on:
- $\text{Spec}(F_0)$, where $F_0$ is the residue field of $F$; and
- $\text{Spec}(L_F)$, from the derived linear order $L_F$ of $F$.

This is false unless we restrict to derived linear orders with no left end point (and allow nonprincipal RCF’s, of course).

Problem: nonprincipal RCF’s in general!