Degrees of Categoricity
of Algebraic Fields

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Computable Categoricity

**Defn.:** A computable structure $\mathcal{A}$ is *computably categorical* if for each computable $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

**Examples:** (Dzgoev, Goncharov; Remmel; Lempp, McCoy, M., Solomon)

- A linear order is computably categorical iff it has only finitely many adjacencies.
- A Boolean algebra is computably categorical iff it has only finitely many atoms.
- An ordered Abelian group is computably categorical iff it has finite rank ($\equiv$ basis as $\mathbb{Z}$-module).
- For trees, the known criterion is recursive in the height and not easily stated!
\textbf{d-Computable Categoricity}

\textbf{Defn.}: For any Turing degree \textit{d}, a computable structure \(A\) is \textit{d-computably categorical} if for each computable \(B \cong A\) there is a \textit{d}-computable isomorphism from \(A\) to \(B\).

\textbf{Example}: \((\omega, \langle\rangle)\) is \(0'\)-computably categorical, although not computably categorical.

\textbf{Defn.}: The \textit{categoricity spectrum} of \(A\) is the set of all \textit{d} such that \(A\) is \textit{d}-computably categorical. The least such degree (if any) is the \textit{degree of categoricity} of \(A\).
Fields

Defn.: The splitting set of a field $F$ is

$$\{p(X) \in F[X] : \exists q_0, q_1 \in F[X](q_0 \cdot q_1 = p)\}.$$ 

Facts:
1. The splitting set is Turing-equivalent to the root set

$$\{p(X) \in F[X] : (\exists a \in F)p(a) = 0\}.$$ 

2. For computable algebraic fields $F_0 \cong F_1$, the splitting sets are Turing-equivalent.

Proofs of these facts use Rabin’s Theorem: A computable field $F$ has a splitting algorithm iff $F$ has a computable embedding with computable image in a computable presentation of $\overline{F}$. 
Negative Results

**Theorem:** There exists a computable algebraic field $F$ which is not computably categorical, yet has computable splitting set.

First idea: Build computable fields $F \cong \tilde{F}$ with all cube roots of all primes $p_e$. If $\varphi_{e,s}(\sqrt[3]{p_e}) \downarrow = y$ with $y^3 = \tilde{p}_e$ in $\tilde{F}$, we adjoin a $p$-th root of $\sqrt[3]{p_e}$ in $F$ and a $p$-th root of a cube root $\neq y$ in $\tilde{F}$.

- Choose $p > s$ to ensure that $F$ has computable splitting set.

- Always use distinct primes $p > 3$: adjoining a $p$-th root cannot cause any extraneous $q$-th roots to appear, for prime $q \neq p$.

**Problem:** Adding a $p$-th root of $\sqrt[3]{p_e}$ puts a $p$-th root of every cube root of $p_e$ into $F$!
Theorem: Let $p$ and $d$ be odd primes, with $F = \mathbb{Q}[\sqrt{p}]$, and let $\sigma(\sqrt{p}) = -\sqrt{p}$. Then there exists a polynomial $h(X) \in F[X]$ of degree $d$, with image $h^-(X) \in F[X]$ under $\sigma$, such that:

- each of the splitting fields $K$ and $K^-$ of $h$ and $h^-$ over $F$ has Galois group $S_d$ over $F$; and
- the splitting field of $h$ over $K^-$ also has Galois group $S_d$, as does the splitting field of $h^-$ over $K$.

So, when $\varphi_e(\sqrt{p_e}) \downarrow= \sqrt{\tilde{p}_e}$, we can adjoin a root of $h(X)$ in $F$ and a root of $\tilde{h}^-(X)$ in $\tilde{F}$.

In fact, this gives us more power:

Theorem: There exists a computable algebraic field $F$ which is not even $\emptyset'$-computably categorical.
$F$ Not 0'-Categorical

Build computable fields $F \cong \tilde{F}$ so that $(\forall e)$

$$f(x) = \lim_{s} \varphi_e(x, s)$$

is not an isomorphism.

Basic module for $\varphi_e$: Adjoin $\pm \sqrt{p_e}$ to $F$ and $\tilde{F}$.

• While $\varphi_e(\sqrt{p_e}, s) \neq \pm \sqrt{p_e}$, do nothing.

• If $\varphi_e(\sqrt{p_e}, s) = \sqrt{p_e}$, then adjoin a root of an $h(X)$ to $F$, and a root of $\tilde{h}^{-}(X)$ to $\tilde{F}$.

• If later $\varphi_e(\sqrt{p_e}, s') = -\sqrt{p_e}$, then adjoin a root of $h^{-}(X)$ to $F$, and a root of $\tilde{h}(X)$ to $\tilde{F}$.

Find a new $h(X)$ for $\sqrt{p_e}$, and do the reverse.

So if $\lim_s \varphi_e(\sqrt{p_e}, s)$ converges, then it chooses the wrong value.

And if $\lim_s \varphi_e(\sqrt{p_e}, s)$ diverges, then we satisfy the requirement and still have $F \cong \tilde{F}$. 
Isomorphisms as Paths

Let $F = \{x_0, x_1, \ldots\}$. Find the minimal polynomial $q_i(X_i)$ of $x_i$ over $\mathbb{Q}[x_0, \ldots, x_{i-1}]$. Write $p_i(x_0, \ldots, x_{i-1}, X_i) = q(X_i)$ with $p_i \in \mathbb{Q}[\tilde{X}]$.

**Defn.** The isomorphism tree $I_{F, \tilde{F}}$ is

$$\{\sigma \in \tilde{F}^n : (\forall i < n)p_{i-1}(\sigma(0), \ldots, \sigma(i - 1)) = 0\}.$$  

So each $\sigma \in I_{F, \tilde{F}}$ defines a partial isomorphism $F \to \tilde{F}$. Paths through $I_{F, \tilde{F}}$ correspond to (total) isomorphisms.
Low Basis Theorem

**Theorem** (Jockusch-Soare): If $T$ is a computable subset of $\omega^\omega$ which forms a finite-branching infinite subtree, and

$$s(\sigma) = |\{\text{immediate successors of } \sigma \text{ in } T\}|$$

has degree $s$, then there is a path $f$ through $T$ with $f' \leq_T s'$.

(Such a path $f$ is said to be **low relative to $s$.**) Indeed, for any fixed $s$, there is a single degree $t$ with $t' \leq_T s'$ which computes a path through every such tree.
**d-Computable Categoricity**

Recall: from the splitting set of $F$, we can compute the number of roots of $p_i(\sigma(0), \ldots, \sigma(i - 1), X_i)$ in $\tilde{F}$.

**Theorem**: If $F$ is a computable algebraic field with splitting set $S$, then $F$ is $d$-computably categorical for some Turing degree $d$ with $d' \leq_T S'$.

**Corollary**: Every computable algebraic field with computable splitting set is $d$-computably categorical for some low Turing degree $d$, indeed for any PA-degree. (A *PA-degree* is the degree of a complete extension of Peano arithmetic.)

**Corollary**: Every computable algebraic field is $d$-computably categorical for some Turing degree $d$ with $d' \leq_T 0''$, indeed for any PA-degree relative to $0'$. 
**Degree of Categoricity I**

**Fact** (Jockusch-Soare): Every nonempty $\Pi^0_1$-class contains paths of degrees $c, d$ with $c \wedge d = 0$.

**Proposition**: A computable algebraic field with splitting set $S$ can only have degree of categoricity $\leq_T \deg(S)$.

**Corollary**: A computable algebraic field with computable splitting set cannot have nonzero degree of categoricity.
Theorem (Fokina-M.): For c.e. degrees $c$ and $d$, we have $c \leq_T d$ iff there exists a computable algebraic field $F$ with degree of categoricity $c$ and splitting set of degree $d$.

Proof: Code a c.e. set $C \in c$ into all isomorphisms between $F$ and $\tilde{F}$, by forcing $\sqrt{p_{2e}} \mapsto \sqrt{\tilde{p}_{2e}}$ iff $e \in C$. Code $D \in d$ into the splitting set by adjoining the square roots of $p_{2e+1}$ when/if $e$ enters $D$. 

Degree of Categoricity II
Extending the Results

**Theorem**: All $d$-computable categoricities so far are *uniform*. The same holds for computable fields of characteristic $p$ algebraic over $F_p$.

- When the field has positive finite transcendence degree over $\mathbb{Q}$, the results still hold, but uniformity fails.

- In characteristic $p$, the results hold (non-uniformly) for *separable* algebraic extensions of $F_p(X_1, \ldots, X_n)$.

- For non-separable algebraic extensions of $F_p(X_1, \ldots, X_n)$, these questions remain open.

- Can the same use of trees be applied to other computable algebraic structures?