Degrees of Categoricity of Algebraic Fields

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Slides available at
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Computable Categoricity

Definition
A computable structure $\mathcal{A}$ is *computably categorical* if for each computable $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Examples: (Dzgoev, Goncharov; Remmel; Lempp, McCoy, M., Solomon)

- A linear order is computably categorical iff it has only finitely many adjacencies.
- A Boolean algebra is computably categorical iff it has only finitely many atoms.
- An ordered Abelian group is computably categorical iff it has finite rank ($\equiv$ basis as $\mathbb{Z}$-module).
- For trees (viewed as partial orders), the known criterion is recursive in the height and not easily stated!
**Definition**

For any Turing degree $d$, a computable structure $A$ is $d$-computably categorical if for each computable $B \cong A$ there is a $d$-computable isomorphism from $A$ to $B$.

**Example**

$(\omega, \prec)$ is 0'-computably categorical, although not computably categorical.
**d-Computable Categoricity**

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**Example**
$(\omega, <)$ is $0'$-computably categorical, although not computably categorical.

**Definition**
The *categoricity spectrum* of $\mathcal{A}$ is the set of all $d$ such that $\mathcal{A}$ is $d$-computably categorical. The least such degree (if any) is the *degree of categoricity of $\mathcal{A}$*. 
Fields

Definition

The *splitting set* of a field $F$ is

$$\{ p(X) \in F[X] : \exists \text{ nonconstant } q_0, q_1 \in F[X] (q_0 \cdot q_1 = p) \}.$$ 

Facts:

1. The splitting set is Turing-equivalent to the *root set*

$$\{ p(X) \in F[X] : (\exists a \in F) p(a) = 0 \}.$$ 

2. For computable algebraic fields $F_0 \cong F_1$, the splitting sets are Turing-equivalent.

Proofs of these facts use **Rabin’s Theorem**: A computable field $F$ has a splitting algorithm iff $F$ has a computable embedding with computable image in a computable presentation of $\overline{F}$. 

Negative Results

**Theorem**

There exists a computable algebraic field $F$ which is not computably categorical, yet has computable splitting set.

First idea: Build computable fields $F \cong \tilde{F}$ with both square roots of each prime $p_e$. If $\varphi_{e,s}(\sqrt{p_e}) \downarrow y$ with $y^2 = \tilde{p}_e$ in $\tilde{F}$, we adjoin a $p$-th root of $\sqrt{p_e}$ in $F$ and a $p$-th root of the square root $\neq y$ in $\tilde{F}$.

- Choose $p > s$ to ensure that $F$ has computable splitting set.
- Always use distinct primes $p > 3$: adjoining a $p$-th root cannot cause any extraneous $q$-th roots to appear, for prime $q \neq p$. 
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**Problem**: Adding a $p$-th root of $\sqrt{p_e}$ puts a $p$-th root of the other square root of $p_e$ into $F$ as well.
Solution to the Problem

Proposition

Let $p$ and $d$ be odd primes, with $F = \mathbb{Q}[\sqrt{p}]$, and let $\sigma(\sqrt{p}) = -\sqrt{p}$. Then there exists a polynomial $h(X) \in F[X]$ of degree $d$, with image $h^-(X) \in F[X]$ under $\sigma$, such that:

- each of the splitting fields $K$ and $K^-$ of $h$ and $h^-$ over $F$ has Galois group $S_d$ over $F$; and
- the splitting field of $h$ over $K^-$ also has Galois group $S_d$, as does the splitting field of $h^-$ over $K$.

So, when $\varphi_e(\sqrt{p_e}) \downarrow = \sqrt{\tilde{p}_e}$, we can adjoin a root of $h(X)$ in $F$ and a root of $\tilde{h}^-(X)$ in $\tilde{F}$. 
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So, when $\varphi_e(\sqrt{p_e}) \downarrow = \sqrt{\tilde{p}_e}$, we can adjoin a root of $h(X)$ in $F$ and a root of $\tilde{h}^-(X)$ in $\tilde{F}$. In fact, this gives us more power.

Theorem

There exists a computable algebraic field $F$ which is not even $\emptyset'$-computably categorical.
A field $F$ which is not $0'$-categorical

Build computable fields $F \cong \tilde{F}$ so that $(\forall e)$

$$f(x) = \lim_{s} \varphi_e(x, s)$$

is not an isomorphism.

Basic module for $\varphi_e$: Adjoin $\pm \sqrt{p_e}$ to $F$ and $\tilde{F}$.

- While $\varphi_e(\sqrt{p_e}, s) \neq \pm \sqrt{\tilde{p_e}}$, do nothing.
- If $\varphi_e(\sqrt{p_e}, s) = \sqrt{\tilde{p_e}}$, then adjoin a root of an $h(X)$ to $F$, and a root of $\tilde{h}^-(X)$ to $\tilde{F}$.
- If later $\varphi_e(\sqrt{p_e}, s') = -\sqrt{\tilde{p_e}}$, then adjoin a root of $h^-(X)$ to $F$, and a root of $\tilde{h}(X)$ to $\tilde{F}$. Find a new $h(X)$ for $\sqrt{p_e}$, and do the reverse.

So if $\lim_s \varphi_e(\sqrt{p_e}, s)$ converges, then it chooses the wrong value.

And if $\lim_s \varphi_e(\sqrt{p_e}, s)$ diverges, then we satisfy the requirement and still have $F \cong \tilde{F}$. 
Isomorphisms as Paths

Let $F = \{x_0, x_1, \ldots\}$. Find the minimal polynomial $q_i(X_i)$ of $x_i$ over $\mathbb{Q}[x_0, \ldots, x_{i-1}]$. Write $p_i(x_0, \ldots, x_{i-1}, X_i) = q(X_i)$ with $p_i \in \mathbb{Q}[\tilde{X}]$.

**Definition**

The *isomorphism tree* $I_{F,\tilde{F}}$ is

$$\{ \sigma \in \tilde{F}^n : (\forall i < n) p_{i-1}(\sigma(0), \ldots, \sigma(i-1)) = 0 \}.$$ 

So each $\sigma \in I_{F,\tilde{F}}$ defines a partial isomorphism $F \to \tilde{F}$. Paths through $I_{F,\tilde{F}}$ correspond to (total) isomorphisms.
Theorem (Jockusch-Soare)

If $T$ is a computable subset of $\omega^{<\omega}$ which forms a finite-branching infinite subtree, and

$$s(\sigma) = |\{\text{immediate successors of } \sigma \text{ in } T\}|$$

has degree $s$, then there is a path $f$ through $T$ with $f' \leq_T s'$. (Such a path $f$ is said to be low relative to $s$.)

Indeed, for any fixed $s$, Jockusch and Soare produced a single degree $t$ with $t' \leq_T s'$ which computes a path through every such tree.
**d-Computable Categoricity**

Recall: from the splitting set of $F$, we can compute the number of roots of $p_i(\sigma(0), \ldots, \sigma(i - 1), X_i)$ in $\tilde{F}$.

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**Theorem**

If $F$ is a computable algebraic field with splitting set $S$, then $F$ is $d$-computably categorical for some Turing degree $d$ with $d' \leq_T S'$. 

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If $F$ is a computable algebraic field with splitting set $S$, then $F$ is $d$-computably categorical for some Turing degree $d$ with $d' \leq_T S'$.

**Corollary**

Every computable algebraic field with computable splitting set is $d$-computably categorical for some low Turing degree $d$, indeed for any PA-degree. (A *PA-degree* is the degree of a complete extension of Peano arithmetic.)
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**Corollary**

Every computable algebraic field is $d$-computably categorical for some Turing degree $d$ with $d' \leq_T 0''$, indeed for any PA-degree relative to $0'$. 
**Fact (Jockusch-Soare)**

Every nonempty $\Pi^0_1$-class contains paths of degrees $c, d$ with $c \land d = 0$.

**Proposition**

A computable algebraic field with splitting set $S$ can only have degree of categoricity $\leq_T \deg(S)$.

**Corollary**

A computable algebraic field with computable splitting set cannot have nonzero degree of categoricity.
More about Degrees of Categoricity

**Theorem**

For c.e. degrees \( c \) and \( d \), we have \( c \leq_T d \) iff there exists a computable algebraic field \( F \) with degree of categoricity \( c \) and splitting set of degree \( d \).

**Proof:** Code a c.e. set \( C \in c \) into all isomorphisms between \( F \) and \( \tilde{F} \), by forcing \( \sqrt{p_{2e}} \mapsto \sqrt{\tilde{p}_{2e}} \) iff \( e \in C \). Code \( D \in d \) into the splitting set by adjoining the square roots of \( p_{2e+1} \) when/if \( e \) enters \( D \).
Extending the Results

Theorem

All $d$-computable categoricities so far are uniform. The same holds for computable fields of characteristic $p$ algebraic over $F_p$.

- When the field has positive finite transcendence degree over $\mathbb{Q}$, the results still hold, but uniformity fails.
- In characteristic $p$, the results hold (non-uniformly) for separable algebraic extensions of $F_p(X_1, \ldots, X_n)$.
- For non-separable algebraic extensions of $F_p(X_1, \ldots, X_n)$, these questions remain open.

Isomorphism trees can be applied to other computable algebraic structures. Cf. work of Rebecca Steiner on finite-branching trees (under predecessor) and finite-valence connected graphs; also Hirschfeldt-Khoussainov-Soare on such graphs.
References on Computable Fields

- R. Steiner; Effective algebraicity, submitted for publication.