APPLICATIONS OF THE $L_2$-TRANSFORM TO PDE’S

The goal of this article is to illustrate the applicability of the $L_2$-transform to solving certain partial differential equations. Recall that a function $f : [0, \infty) \to \mathbb{R}$ is called exponential squared order if $\lim_{x \to \infty} f(x)e^{-x^2} = 0$.

**Definition 1.** For any exponential squared order function $f(t)$, the $L_2$ transform of $f$ is defined as:

$$L_2\{f(x); s\} = \int_0^\infty xe^{-x^2s^2}f(x)dx$$

Here is a useful example. For $n \geq 0$ we have:

(1) $L_2\{x^{2n}; s\} = \frac{n!}{2^{n^2 + 2}}$

In the case $n = 0$, we obtain

$$L_2\{1; s\} = \int_0^\infty xe^{-x^2s^2}dx = \frac{1}{2s^2}$$

and for $n = 1$, through integration by parts, we have

$$L_2\{x^2; s\} = \left(\frac{-x^2}{2s^2}e^{-x^2s^2}\right)|^\infty_0 + \frac{1}{s^2}\int_0^\infty e^{-x^2s^2}xdx = \frac{1}{s^2}\left(\frac{1}{2s^2}\right)$$

Property (1) then follows by a simple induction.

There is a differential operator $\delta_x$, defined by

$$\delta_x = \frac{1}{x} \cdot \frac{d}{dx}$$

which has some nice properties with respect to the $L_2$-transform, which we now recall. See also

**Proposition 1.** Let $f$ be a function of exponential squared order. Then

(2) $L_2\{\delta_x f(x); s\} = 2s^2L_2\{f(x); s\} - f(0^+)$

and for all $n \geq 0$

(3) $L_2\{x^{2n}f(x); s\} = \frac{(-1)^n}{2^n}\delta^n_x L_2\{f(x); s\}$

**Proof.** For the first claim, we calculate

$$\int_0^\infty xe^{-x^2s^2}\delta_x f(x)dx = \int_0^\infty xe^{-x^2s^2} \cdot \frac{1}{x} \cdot \frac{d}{dx} f(x)dx = \int_0^\infty e^{-x^2s^2} f'(x)dx$$

and, integrating by parts, we get:

$$= f(x)(e^{-x^2s^2})|^\infty_0 + \int_0^\infty 2xs^2xe^{-x^2s^2} f(x)dx$$

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Evaluating from 0 to ∞, using the fact that f is exponential squared order, we can write this expression in terms of the $L_2$ transform:

$$2s^2L_2\{f(x); s\} - f(0^+)$$

For property (3), taking the case in which $n=1$, we get:

$$\delta_n L_2\{f(x); s\} = \delta_n \int_0^\infty xe^{-x^2s^2} f(x)dx$$

We can take the differential operator $\delta_n$ into the integral, yielding:

$$\int_0^\infty \frac{1}{s} \frac{d}{ds} xe^{-x^2s^2} f(x)dx = \int_0^\infty \frac{1}{s} - 2x^2sxe^{-x^2s^2} f(x)dx = -2 \int_0^\infty x^2xe^{-x^2s^2} f(x)dx$$

We can notice here that the last term of this equation is really equal to:

$$L_2\{-2x^2f(x); s\}$$

Which is an alternative representation of property (3). Applying the application of the differential operator with respect to $s$ to this result, we get:

$$\delta_n L_2\{-2x^2f(x); s\} = \delta_n - 2 \int_0^\infty x^2xe^{-x^2s^2} f(x)dx$$

As we did before, we re-write the differential operator and apply it within the integral:

$$-2 \int_0^\infty \frac{1}{s} \frac{d}{ds} x^2xe^{-x^2s^2} f(x)dx = -2 \int_0^\infty \frac{1}{s} x^2x^2xe^{-x^2s^2} f(x)dx - 2x^2sdx = 4 \int_0^\infty x^2x^2xe^{-x^2s^2} f(x)dx$$

Which we can see is really equal to:

$$4L_2\{x^4f(x); s\}$$

As we can see through induction, for a general $n \geq 1$, we have:

$$\delta_n L_2\{f(x); s\} = -2^n L_2\{x^{2n}f(x); s\}$$

Or, written alternatively,

$$L_2\{x^{2n}f(x); s\} = \frac{(-1)^n}{2^n} \delta_n L_2\{f(x); s\}$$

\[\square\]

1. **The $L_2$ Convolution**

A binary operation $(\ast)$ called the convolution of two functions $f, g$ is defined as follows:

$$(f \ast g)(t) = \int_0^t xf(\sqrt{t^2 - x^2})g(x)dx$$

It can be shown that this operation is associative and commutative. Most importantly for our purposes, the following are true:

$$L_2\{f \ast g; s\} = L_2(f) \cdot L_2(g)$$

and:

$$L_2\{f_1 \ast f_2 \ast \ldots \ast f_n; s\} = L_2(f_1) \cdot L_2(f_2) \cdot \ldots \cdot L_2(f_n)$$

From which it follows:

$$L^{-1}\{\hat{f}_1 \cdot \hat{f}_2 \cdot \ldots \cdot \hat{f}_n\} = f_1 \ast f_2 \ast \ldots \ast f_n$$
2. Applications

Consider the following partial differential equation:

\[(6) \quad t^3 u_{tx} + 2xu = 0 \quad \text{and} \quad u(0^+, t) = 0\]

where \( u(x, t) \) for \( x, t > 0 \) and \( u(0^+, t) = \lim_{x \to 0^+} u(x, t) \).

Writing the PDE in Equation 6 in terms of the differential operator, we have:

\[t^3 \frac{1}{x} \frac{d}{dx} u_t + 2u = 0\]

or equivalently,

\[t^3 \delta_x u_t = -2u.\]

Taking the \( L_2 \) transform of both sides and using property (3) we obtain

\[2s^2 t^3 \hat{u}_t - u(0^+, t) = -2\hat{u} \]

where \( \hat{u} = \hat{u}(s, t) = \mathcal{L}_2^{-1}(u(x, t); s) \). Equivalently,

\[\hat{u}_t = -\frac{1}{s^2 t^3} \hat{u} + \frac{u(0^+, t)}{2s^2 t^3}.\]

By the initial condition in (6), this last term is zero. One solution to this differential equation is

\[\hat{u}(s, t) = \frac{1}{2s^2} e^{-\frac{t^2-x^2}{4}}\]

We now write \( \hat{u}(s, t) \) as a series to obtain:

\[\hat{u}(s, t) = \frac{1}{2s^2} \sum_{n=0}^{\infty} \frac{(\frac{x^2}{2})^{-2n}}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{2s^2(n)!^2}.\]

Using property (1), we can calculate that \( u(x, t) = \mathcal{L}_2^{-1}\{\hat{u}(s, t); x\} \) is

\[ u(x, t) = \sum_{n=0}^{\infty} \frac{2^n x^{2n}}{(n!)^2}.\]

3. Further Generalizations

Similarly, we can solve all partial differential equations of the form:

\[0 = f(t)u + f(t) \frac{1}{x} u_x + g(t) \frac{1}{x} u_{xt}\]

Taking the \( L_2 \) transform, we compute:

\[0 = f(t)\hat{u} + f(t)2s^2 \hat{u} + g(t)2s^2 \hat{u}_t\]

Rearranging:

\[\frac{\hat{u}_t}{\hat{u}} = -\frac{(1 + 2s^2)}{2s^2} \cdot M(t)\]

Where \( M(t) = \frac{f(t)}{g(t)} \). Claiming that \( L(s) \) is any function of \( s \), a solution for \( \hat{u} \) is as follows:

\[\hat{u}(s, t) = e^{-\frac{(1 + 2s^2)}{2s^2} \int_0^t M(w)dw} \cdot L(s) = e^{\int_0^t M(w)dw} \cdot e^{-\frac{\int_0^t M(w)dw}{2s^2}} \cdot L(s)\]
Writing this as a series, denoting $\int_t^0 M(w)dw$ as $M$, and letting $L(s) = \frac{1}{-s^2}$, we get:

$$\hat{u} = \frac{1}{-s^2} \sum_{n=0}^{\infty} \frac{-M^n}{n!} \sum_{n=0}^{\infty} \frac{M^n}{n!} = \sum_{n=0}^{\infty} \frac{M^n}{n!} \cdot (2s^{n+2}) \cdot \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

using identity (2), and taking the $L_2$ inverse transform, we claim that:

$$u(x,t) = \sum_{n=0}^{\infty} M^n \cdot x^{2n} \cdot (2n) \cdot \sum_{n=0}^{\infty} \frac{M^n}{n!} = \sum_{n=0}^{\infty} M^n \cdot 2n \cdot x^{2n} \cdot \sum_{n=0}^{\infty} \frac{M^n}{n!} \cdot (2n)$$

Lastly, we can always solve the following types of partial differential equations:

$$0 = f(t)u + f(t)\frac{1}{x}u_x + g(t)u_t + g(t)u + xt$$

Taking the $L_2$ transform, we get:

$$0 = \hat{u} f(t) + \hat{u}_t = \frac{\hat{u}_t}{\hat{u}} = J(t)$$

Where $J(t) = \frac{g(t)}{f(t)}$. This implies that a solution for $\hat{u}$ is as follows:

$$\hat{u}(s,t) = e^{\int_t^0 J(w)dw} \cdot L(s)$$

4. An Application to a Partial Differential Equation of Exponential Squared Order

Consider the following partial differential equation, with the following condition:

$$0 = g(t)u - f(t)u_t + \frac{1}{x} f(x)u_{xt} \quad u(0^+, t) = 0$$

Writing this equation in terms of the differential operator,

$$0 = g(t)u - f(t)u_t + \delta_x f(t)$$

and taking the $L_2$ transform of it, we get:

$$0 = g(t)\hat{u}_t - f(t)\hat{u}_t + 2s^2 f(t)\hat{u}_t$$

Rearranging, we get:

$$0 = g(t)\hat{u}_t - f(t)\hat{u}_t = 2s^2 f(t)\hat{u}_t$$

Which leads to:

$$\frac{\hat{u}_t}{\hat{u}} = -\frac{1}{-1 + 2s^2} \cdot H(t)$$

Where $H(t) = \frac{g(t)}{f(t)}$. This implies that a solution for $\hat{u}$ is as follows:

$$\hat{u}(s,t) = e^{-\int_t^0 \frac{1}{1 + 2s^2} H(w)dw}$$
Writing this solution as a series, we get:
\[ \hat{u}(s, t) = \sum_{n=0}^{\infty} \left( \frac{\int_0^t H(w)dw}{n!} \right)^n \left( \frac{1}{-1 + 2s^2} \right)^n \]
from (4), (5) and the following identity:
\[ L_2\{e^{ax^2}; s\} = \frac{1}{-2a + 2s^2} \]
We can arrive at the following solution:
\[ L^{-1}_2(\hat{u}(s, t)) = \sum_{n=1}^{\infty} \left( \frac{\int_0^t H(w)dw}{n!} \cdot (e^{(1/2)x^2})^n \right) \]
Where \( e^{(1/2)x^2} \) represents the following:
\[ e^{(1/2)x^2} \times e^{(1/2)x^2} \times \ldots \times e^{(1/2)x^2} \]
Which leads the following proposition, which can be shown through induction:
\[ (e^{(1/2)x^2})^n = \frac{1}{2^{n-1}} \cdot \frac{x^{2(n-1)} \cdot (e^{(1/2)x^2})}{(n-1)!} \]
After a substitution, we have the following solution for \( u(x, t) \):
\[ u(x, t) = \sum_{n=1}^{\infty} \left( \frac{\int_0^t H(w)dw}{n!} \cdot \frac{1}{2^{n-1}} \cdot \frac{x^{2(n-1)} \cdot (e^{(1/2)x^2})}{(n-1)!} \right) \]
Notice that the \( e^{(1/2)x^2} \) term in this sum prohibits this solution to satisfy the definition of exponential order. To find the limit of this solution when multiplied with \( e^{-x^2} \) as \( x \to \infty \), we recognize the following:
\[ \sum_{n=0}^{\infty} \frac{x^{2n} \cdot (e^{(1/2)x^2})}{(n+1)(n!)^2} = e^{x^2/2} \sum_{n=0}^{\infty} \frac{(\frac{x^2}{2})^n}{n!} \leq e^{x^2/2} \sum_{n=0}^{\infty} \frac{(\frac{x^2}{4})^n}{n!} = e^{x^2/2} \cdot e^{x^2/4} \]
Which follows from the fact that:
\[ (n+1)(n!) \leq 2^n \quad \forall \ n \geq 0 \]
And since the following is true:
\[ \lim_{x \to \infty} e^{-x^2} \cdot e^{x^2/2} \cdot e^{x^2/4} = 0 \]
We see that our solution is in fact, of exponential squared order. Thus, this is a solution that could not have been obtained through some other integral transforms, including the Fourier transform.

**References**