COCHAIN ALGEBRA ON MANIFOLDS AND CONVERGENCE UNDER REFINEMENT

SCOTT O. WILSON

Abstract. In this paper we develop several algebraic structures on the simplicial cochains of a triangulated manifold and prove they converge to their differential-geometric analogues as the triangulation becomes small. The first such result is for a cochain cup product converging to the wedge product on differential forms. Moreover, we show any extension of this product to a $\mathcal{C}_\infty$-algebra also converges to the wedge product of forms. For cochains equipped with an inner product, we define a *combinatorial star operator* and show that for a certain cochain inner product this operator converges to the smooth Hodge star operator.

1. Introduction

In this paper we develop combinatorial analogues of several objects in differential geometry, including the Hodge star operator. We define these structures on the appropriate combinatorial analogue of differential forms, namely simplicial cochains.

As we recall in section 3, two ingredients for defining the smooth Hodge star operator are Poincaré Duality and a metric, or inner product. In a similar way, we define a *combinatorial star operator* using an inner product and Poincaré Duality, the latter expressed on simplicil cochains in the form of a graded commutative (non-associative) product.

Using the inner product introduced in [9], we prove the following:

Theorem 1.1. *The combinatorial star operator, defined on the simplicial cochains of a triangulated Riemannian manifold, converges to the smooth Hodge star operator as the mesh of the triangulation tends to zero.*

This convergence statement is made precise by using an embedding of simplicial cochains into differential forms, first introduced by Whitney [33]. This approach was used quite successfully by Dodziuk [9], and later Dodziuk and Patodi [10], to show that cochains provide a good approximation to smooth differential forms, and that the combinatorial Laplacian converges to the smooth Laplacian. This formalism will be reviewed in section 4.

In section 5 we describe the cochain product we use to define the combinatorial star operator. This product is of interest in its own right, and we prove several results concerning its convergence to the wedge product on forms. We expect these result may be of interest in numerical analysis and the modeling of PDE's, since

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they give a computable discrete model which approximates the algebra of smooth differential forms.

In section 6 we introduce the combinatorial star operator, and prove our covergence theorem. We also study, in the combinatorial setting, many of the interesting relations among $\star$, d, $\wedge$, and the adjoint d^* of d, that hold in the smooth setting.

In the last two sections, 7 and 8, we show how an explicit computation of the combinatorial star operator is related to "summing over weighted paths," and perform these calculations for the circle.

We should remark, one motivation for this work comes from statistical mechanics [7], where one would like to have a lattice-version of differential geometry. In a separate work we show that, on a closed surface, the combinatorial star operator defined here gives rise to a combinatorial period matrix. Furthermore, the combinatorial period matrix of a triangulated Riemannian 2-manifold converges to the conformal period matrix of the associated Riemann surface, as the mesh of the triangulation tends to zero. This suggests a link between statistical mechanics and conformal field theory, where it is known that the partition function may be expressed in terms of theta functions of the conformal period matrix [21], see also [7], [23].

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2. Background and Acknowledgments

In this section we describe previous results that are related to the contents of this paper. My sincere apologies to anyone whose work I have left out.

The cochain product we discuss was introduced by Whitney in [33]. It was also studied by Sullivan in the context of rational homotopy theory [29], by DuPont in his study of curvature and characteristic class [12], and by Birmingham and Rakowski as a star product in lattice gauge theory [5].

In connection with our result on the convergence of this cochain product to the wedge product of forms, Kervaire has a related result for the Alexander-Whitney product $\cup$ on cochains [20]. Kervaire states that, for differential forms A, B, and the associated cochains a, b,

$$\lim_{k\to\infty} a \cup b\ (S^k c) = \int_c A \wedge B$$

for a convenient choice of subdivisions $S^k c$ of the chain c. Cheeger and Simons use this result in the context of cubical cell structures in [6]. There they construct an explicit map $E(A, B)$ satisfying

$$\int A \wedge B - a \cup b = \delta E(A, B)$$

and use it in the development of the theory of differential characters. To the best of our knowledge, our convergence theorems for the commutative cochain product in section 5 are the first to appear in the literature.

Several definitions of a discrete analogue of the Hodge-star operator have been made. In [7], Costa-Santos and McCoy define a discrete star operator for a particular 2-dimensional lattice and study convergence properties as it relates to the Ising Model. Mercat defines a discrete star operator for surfaces in [22], using a triangulation and its dual, and uses it to study a notion of discrete holomorphy and its relation to Ising criticality.

In [30], Tarhasaari, Kettunen and Bossavit describe how to make explicit computations in electromagnetism using Whitney forms and a star operator defined using the de Rham map from forms to cochains. Teixeira and Chew [31] have also defined Hodge operators on a lattices for the purpose of studying electromagnetic theory.

Adams [1], and also Sen, Sen, Sexton and Adams [25], define two discrete star operators using a triangulation and its dual, and present applications to lattice gauge fields and Chern-Simons theory. De Beaucé and Sen [3] define star operators in a similar way and study applications to chiral Dirac fermions; and de Beaucé and Sen [4] have generalized this to give a discretization scheme for differential geometry [4].

Desbrun, Hirani, Leok and Marsden have presented a "discrete exterior calculus" in [8]. There, the authors work on the simplicial cochains of a triangulation and its dual, and give a theory that includes exterior d, a wedge product, a Hodge star, a Lie derivative and contraction on vector fields, as well as several applications.

In the approaches using a triangulation and its dual, the star operator(s) are formulated using the duality map between the two cell decompositions. This map yields Poincaré Duality on (co)homology. By contrast, we express Poincaré Duality by a commutative cup product on cochains and combine it with a non-degenerate inner product to define the star operator. Working this way, we obtain a single operator from one complex to itself.

Our convergence statements in section 6 are proven using the inner product introduced in [9], and to the best of our knowledge, these are the first results proving a convergence theorem for a cochain-analogue of the Hodge star operator.

In Dodziuk's paper [9], and in [10] by Dodziuk and Patodi, the authors study a combinatorial Laplacian on the cochains and proved that its eigenvalues converge to the smooth Laplacian. Such discrete notions of a Hodge structure, along with finite element method techniques, were used by Kotiuga [19], and recently by Gross and Kotiuga [14], in the study of computational electromagnetism. Jin has used related techniques in studying electrodynamics [15].

Harrison's development of 'chainlet geometry' in [16], [17], and [18] has several themes similar to those in this paper. In her new approach to geometric measure theory, the author develops 'dual analogues' of d, $\wedge$ and $\star$ by defining them on chainlets, a Banach space defined by taking limits of polyhedral chains. Chainlets are, in a sense, dual to differential forms in that they are 'domains of integration'. The author proves several convergence results for these analogues, and it appears these constructions and results have many applications as well.

3. Smooth Setting

We begin with a brief review of some elementary definitions. Let M be a closed oriented Riemannian n-manifold. A Riemannian metric induces an inner product on $\Omega(M) = \bigoplus_j \Omega^j = \bigoplus_j \Gamma(\bigwedge^j T^*M)$ in the following way: a Riemannian metric determines an inner product on T^*M_p for all p, and hence an inner product for each j on $\bigwedge^j T^*M_p$ (explicitly, via an orthonormal basis). An inner product $\langle , \rangle$ on $\Omega(M)$ is then obtained by integration over M. If we denote the induced norm on $\bigwedge^j T^*M_p$ by $| \quad |_p$, then the norm $\| \quad \|$ on $\Omega(M)$ is given by

$$\|\omega\| = \Big(\int_M |\omega|_p^2 \; dV \Big)^{1/2}$$

where dV is the Riemannian volume form on M.

Let $\mathcal{L}_2\Omega(M)$ denote the completion of $\Omega(M)$ with respect to this norm. We also use $\| \quad \|$ to denote the norm on the completion. Let the exterior derivative $d : \Omega^j(M) \to \Omega^{j+1}(M)$ be defined as usual.

Definition 3.1. *The Poincare-Duality pairing* $(,) : \Omega^j(M) \otimes \Omega^{n-j}(M) \to \mathbb{R}$ *is defined by:*

$$(\omega, \eta) = \int_M \omega \wedge \eta.$$

The pairing $(,)$ is bilinear, (graded) skew-symmetric and non-degenerate. It induces an injection $\phi : \Omega^j(M) \to (\Omega^{n-j}(M))^*$, where here $*$ denotes the linear dual. The map $\psi : \Omega^{n-j}(M) \to (\Omega^{n-j}(M))^*$ induced by $\langle , \rangle$ is an isomorphism and one may check that the composition $\psi^{-1} \circ \phi$ equals the following operator:

Definition 3.2. *The Hodge star operator* $\star : \Omega^j(M) \to \Omega^{n-j}(M)$ *is defined by:*

$$\langle \star\omega, \eta \rangle = (\omega, \eta).$$

One may also define the operator $\star$ using local coordinates, see Spivak [27]. We note that this approach and the former definition give rise to the same operator $\star$ on $\mathcal{L}_2\Omega(M)$. We prefer to emphasize definition 3.2 since it motivates definition 6.1, the combinatorial star operator.

Definition 3.3. *The adjoint of d, denoted by d^*, is defined by* $\langle d^*\omega, \eta \rangle = \langle \omega, d\eta \rangle$.

Note that $d^* : \Omega^j(M) \to \Omega^{j-1}(M)$. The following relations hold among $\star$, d and d^*. See Spivak [27].

Theorem 3.4. *As maps from $\Omega^j(M)$ to their respective ranges:*

1. $\star d = (-1)^{j+1} d^* \star$
2. $\star d^* = (-1)^j d \star$
3. $\star^2 = (-1)^{j(n-j)}$ Id

Definition 3.5. *The Laplacian is defined to be* $\Delta = d^*d + dd^*$.

Finally, we state the Hodge decomposition theorem for $\Omega(M)$. Let $\mathcal{H}^j(M) = \{\omega \in \Omega^j(M) | \Delta\omega = 0\}$ be the space of harmonic j-forms. Recall that $\Delta\omega = 0$ if and only if $d\omega = d^*\omega = 0$.

Theorem 3.6. *There is an orthogonal direct sum decomposition*

$$\Omega^j(M) \cong d\Omega^{j-1}(M) \oplus \mathcal{H}^j(M) \oplus d^*\Omega^{j+1}(M)$$

and $\mathcal{H}^j(M) \cong H^j_{DR}(M)$*, the De Rham cohomology of M in degree j.*

4. Whitney Forms

In his book, 'Geometric Integration Theory', Whitney explores the idea of using cochains as integrands [33]. A main result is that such objects provide a reasonable integration theory that in some sense generalizes the smooth theory of integration of differential forms. This idea has been made even more precise by the work of Dodziuk [9], who used a linear map of cochains into piecewise linear forms (due to Whitney [33]) to show that cochains provide a good approximation of differential forms. In this section we review some of these results. The techniques involved illustrate a tight (and analytically precise) connection between cochains and forms, and will be used later to give precise meaning to our constructions on cochains. In particular, all of our convergence statements about combinatorial and smooth objects will be cast in a similar way.

Let M be a closed smooth n-manifold and K a fixed C^∞ triangulation of M. We identify $|K|$ and M and fix an ordering of the vertices of K. Let C^j denote the simplicial cochains of degree j of K with values in $\mathbb{R}$. Given the ordering of the vertices of K, we have a coboundary operator $\delta : C^j \to C^{j+1}$. Let μ_i denote the barycentric coordinate corresponding to the i^{th} vertex p_i of K. Recall the barycentric coordinates $\mu_0, \dots, \mu_j$ determine the j-simplex by the equations $0 \leq \mu_i \leq 1$ and $\sum_{i=0}^{j} \mu_i = 1$. Since M is compact, we may identify the cochains and chains of K and for $c \in C^j$ write $c = \sum_\tau c_\tau \cdot \tau$ where $c_\tau \in \mathbb{R}$ and is the sum over all j-simplices τ of K. We write $\tau = [p_0, p_1, \dots, p_j]$ of K with the vertices in an increasing sequence with respect to the ordering of vertices in K. We now define the Whitney embedding of cochains into $\mathcal{L}_2$-forms:

Definition 4.1. *For τ as above, we define*

$$W\tau = j! \sum_{i=0}^{j} (-1)^i \mu_i \, d\mu_0 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_j.$$

W is defined on all of C^j by extending linearly.

Note that the coordinates μ_α are not even of class C^1, but they are C^∞ on the interior of any n-simplex of K. Hence, $d\mu_\alpha$ is defined and $W\tau$ is a well defined element of $\mathcal{L}_2\Omega^j$. By the same consideration, dW is also well defined. Note both sides of the definition of W are alternating, so this map is well defined for all simplices regardless of the ordering of vertices.

Several properties of the map W are given below. See [33],[9], [10] for details.

Theorem 4.2. *The following hold:*

1. $W\tau = 0$ *on* $M \setminus \overline{St(\tau)}$
2. $dW = W\delta$

where $St(\tau)$ denotes the open star of τ, i.e. the interior of all simplices whose closure contain τ, and $\overline{}$ denotes closure.

One also has a map $R : \Omega^j(M) \to C^j(K)$, the de Rham map, given by integration. Precisely, for any differential form ω and chain c we have:

$$R\omega(c) = \int_c \omega$$

It is a theorem of de Rham that this map is a quasi-isomorphism (it is a chain map by Stokes Theorem). RW is well defined and one can check that $RW = Id$, see [33], [9], [10].

Before stating Dodziuk and Patodi's theorem that WR is approximately equal to the identity, we first give some definitions concerning triangulations. They also appear [10].

Definition 4.3. *Let K be a triangulation of an n-dimensional manifold M. The mesh $\eta = \eta(K)$ of a triangulation is:*

$$\eta = \sup r(p, q),$$

where r means the geodesic distance in M and the supremum is taken over all pairs of vertices p, q of a 1-simplex in K.

The fullness $\Theta = \Theta(K)$ of a triangulation K is

$$\Theta(K) = \inf \frac{vol(\sigma)}{\eta^n},$$

where the inf *is taken over all n-simplexes σ of K and $vol(\sigma)$ is the Riemannian volume of σ, as a Riemannian submanifold of M.*

A Euclidean analogue of the following lemma was proven by Whitney in [33] (IV.14).

Lemma 4.4. *Let M be a smooth Riemannian n-manifold.*

1. *Let K be a smooth triangulation of M. Then there is a positive constant $\Theta_0 > 0$ and a sequence of subdivisions $K_1, K_2, \ldots$ of K such that $\lim_{n\to\infty} \eta(K_n) = 0$ and $\Theta(K_n) \geq \Theta_0$ for all n.*
2. *Let $\Theta_0 > 0$. There exist positive constants C_1, C_2 depending on M and Θ_0 such that for all smooth triangulations K of M satisfying $\Theta(K) \geq \Theta_0$, all n-simplexes of $\sigma = [p_0, p_1, \ldots, p_n]$ and vertices p_k of σ,*

$$\begin{aligned} vol(\sigma) &\leq C_1 \cdot \eta^n \\ C_2 \cdot \eta &\leq r(p_k, \sigma_{p_k}), \end{aligned}$$

where r is the Riemannian distance, $vol(\sigma)$ is the Riemannian volume, and $\sigma_{p_k} = [p_0, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n]$ is the face of σ opposite to p_k.

Since any two metrics on M are commensurable, the lemma follows from Whitney's Euclidean result, see also [10].

We consider only those triangulations with fullness bounded below by some positive real constant Θ_0. By the lemma, this guarantees that the volume of a simplex is on the order of its mesh raised to the power of its dimension. Geometrically, this means that in a sequence of triangulations, the shapes do not become too thin. (In fact, Whitney's *standard subdivisions* yield only finitely many shapes, and can be used to prove the first part of the lemma.) Most of our estimates depend on Θ_0, as can be seen in the proofs. We'll not indicate this dependence in the statements.

The following theorems are proved by Dodziuk and Patodi in [10]. They show that for a fine triangulation, WR is approximately equal to the identity. In this sense, the theorems give precise meaning to the statement: for a fine triangulation, cochains provide a good approximation to differential forms.

Theorem 4.5. *Let ω be a smooth form on M, and σ be an n-simplex of K. There exists a constant C, independent of ω, K and σ, such that*

$$|\omega - WR\omega|_p \leq C \cdot sup\left|\frac{\partial \omega}{\partial x^i}\right| \cdot \eta$$

for all $p \in \sigma$. The supremum is taken over all $p \in \sigma$ and $i = 1, 2, \ldots n$, and the partial derivatives are taken with respect to a coordinate neighborhood containing σ.

Proof. A generalization of this theorem will be proved in this paper; see theorem 5.4 and remark 5.5. □

By integrating the above point-wise and applying a Sobolev inequality, Dodziuk and Patodi [10] obtain the following

Corollary 4.6. *There exist a positive constant C and a positive integer m, independent of K, such that*

$$\|\omega - WR\omega\| \leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta$$

for all C^∞ j-forms ω on M.

Proof. This is a special case of corollary 5.7. □

Now suppose the cochains $C(K)$ are equipped with a non-degenerate inner product $\langle , \rangle$ such that, for distinct i, j, $C^i(K)$ and $C^j(K)$ are orthogonal. Then one can define further structures on the cochains. In particular, we have the following

Definition 4.7. *The adjoint of δ, denoted by δ^*, is defined by $\langle \delta^* \sigma, \tau \rangle = \langle \sigma, \delta \tau \rangle$.*

Note that $\delta^* : C^j(K) \to C^{j-1}(K)$ is also squares to zero. One can also define

Definition 4.8. *The combinatorial Laplacian is defined to be $\blacktriangle = \delta^*\delta + \delta\delta^*$.*

Clearly, both δ^* and $\blacktriangle$ depend upon the choice of inner product. For any choice of positive definite inner product, these operators give rise to a combinatorial Hodge theory: the space of harmonic j-cochains of K is defined to be

$$\mathcal{H}C^j(K) = \{a \in C^j | \blacktriangle a = 0\}.$$

As in the smooth case, $\blacktriangle a = 0$ if and only if $\delta a = \delta^* a = 0$. The following theorem is due to Eckmann [13]:

Theorem 4.9. *Let (C, δ) be a finite dimensional complex with inner product $\langle , \rangle$ and induced adjoint δ^* as above. There is an orthogonal direct sum decomposition*

$$C^j(K) \cong \delta C^{j-1}(K) \oplus \mathcal{H}C^j(K) \oplus \delta^* C^{j+1}(K)$$

and $\mathcal{H}C^j(K) \cong H^j(K)$, the cohomology of (K, δ) in degree j.

Proof. The second statement of the theorem follows from the first. We'll write C^j for $C^j(K)$, and let δ^*_j denote δ^* restricted to C^j.

Since δ^* is the adjoint of δ, the orthogonal complement of δC^{j-1} in C^j equals $Ker(\delta^*_j)$. That is, $C^j = \delta C^{j-1} \oplus Ker(\delta^*_j)$ for all j.

Also, since $\delta\delta = \delta^*\delta^* = 0$, it is easy to check that δC^{j-1}, $\mathcal{H}C^j$, and $\delta^* C^{j+1}$ are orthogonal. Since $Ker(\delta^*_j) \supset \mathcal{H}C^j \oplus \delta^* C^{j+1}$, it suffices to show

$$\dim Ker(\delta^*_j) = \dim \mathcal{H}C^j + \dim \delta^* C^{j+1}$$

By considering the restriction of δ to $Ker(\delta_j^*)$ we have that

$$\dim Ker(\delta_j^*) - \dim \mathcal{H}C^j = \dim \delta Ker(\delta_j^*) = \dim \delta C^j$$

The proof is complete by showing $\dim \delta^* C^{j+1} = \dim \delta C^j$. This holds because, by the adjoint property, both $\delta : \delta^* C^{j+1} \to \delta C^j$ and $\delta^* : \delta C^j \to \delta^* C^{j+1}$ are injections of finite dimensional vector spaces. □

If K is a triangulation of a Riemannian manifold M, then there is a particularly nice inner product on $C(K)$, which we'll call the *Whitney inner product*. It is induced by the metric $\langle , \rangle$ on $\Omega(M)$ and the Whitney embedding of cochains into $\mathcal{L}_2$-forms. We'll use the same notation $\langle , \rangle$ for this pairing on C: $\langle \sigma, \tau \rangle = \langle W\sigma, W\tau \rangle$.

It is proven in [9] that the Whitney inner product on C is non-degenerate. Further consideration of this inner product will be given in later sections. For now, following [9] and [10], we describe how the combinatorial Hodge theory, induced by the Whitney inner product, is related to the smooth Hodge theory. Precisely, we have the following theorem due to Dodziuk and Patodi [10], which shows that the approximation $WR \approx Id$ respects the Hodge decompositions of $\Omega(M)$ and $C(K)$.

Theorem 4.10. *Let $\omega \in \Omega^j(M)$, $R\omega \in C^j(K)$ have Hodge decompositions*

$$\begin{aligned} \omega &= d\omega_1 + \omega_2 + d^*\omega_3 \\ R\omega &= \delta a_1 + a_2 + \delta^* a_3 \end{aligned}$$

Then,

$$\begin{aligned} \|d\omega_1 - W\delta a_1\| &\leq \lambda \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \\ \|\omega_2 - W a_2\| &\leq \lambda \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \\ \|d^*\omega_3 - W\delta^* a_3\| &\leq \lambda \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \end{aligned}$$

where λ and m are independent of ω and K.

5. Cochain Product

In this section we describe a commutative, but non-associative, cochain product. It is of interest in its own right, and will be used to define the combinatorial star operator.

The product we define is induced by the Whitney embedding and the wedge product on forms, but also has a nice combinatorial description. An easy way to state this is as follows: the product of a j-simplex and k-simplex is zero unless these simplices span a common $(j + k)$-simplex, in which case the product is a rational multiple of this $(j + k)$-simplex. We will prove a convergence theorem for this product, and also show that this product's deviation from being associative converges to zero for 'sufficiently smooth' cochains.

From the point of view of homotopy theory, it is natural to consider this commutative cochain product as part of a $\mathcal{C}_\infty$-algebra. We use Sullivan's local construction of a $\mathcal{C}_\infty$-algebra [32], and show that this structure converges to the strictly commutative associative algebra given by the wedge product on forms. In particular, all of the higher homotopies of the $\mathcal{C}_\infty$-algebra converge to zero. Only definition 5.1 and theorem 5.2 are used in later sections.

We begin with the definition of a cochain product on the cochains of a fixed triangulation K.

Definition 5.1. *We define* $\cup : C^j(K) \otimes C^k(K) \to C^{j+k}(K)$ *by:*

$$\sigma \cup \tau = R(W\sigma \wedge W\tau)$$

Since R and W are chain maps with respect to d and δ, it follows that δ is a derivation of $\cup$, that is, $\delta(\sigma \cup \tau) = \delta\sigma \cup \tau + (-1)^{deg(\sigma)}\sigma \cup \delta\tau$. Also, since $\wedge$ is graded commutative, $\cup$ is as well: $\sigma \cup \tau = (-1)^{deg(\tau)deg(\sigma)}\tau \cup \sigma$. It follows from a theorem of Whitney [34] that the product $\cup$ induces the same map on cohomology as the usual (Alexander-Whitney) simplicial cochain product. We now give a combinatorial description of $\cup$, this also appears in [2].

Theorem 5.2. *Let* $\sigma = [p_{\alpha_0}, p_{\alpha_1}, \ldots, p_{\alpha_j}] \in C^j(K)$ *and* $\tau = [p_{\beta_0}, p_{\beta_1}, \ldots, p_{\beta_k}] \in C^k(K)$. *Then* $\sigma \cup \tau$ *is zero unless* σ *and* τ *intersect in exactly one vertex and span a* $(j+k)$*-simplex* υ, *in which case, for* $\tau = [p_{\alpha_j}, p_{\alpha_{j+1}}, \ldots, p_{\alpha_{j+k}}]$, *we have:*

$$\begin{aligned} \sigma \cup \tau &= [p_{\alpha_0}, p_{\alpha_1}, \ldots, p_{\alpha_j}] \cup [p_{\alpha_j}, p_{\alpha_{j+1}}, \ldots, p_{\alpha_{j+k}}] \\ &= \epsilon(\sigma,\tau)\frac{j!k!}{(j+k+1)!}[p_{\alpha_0}, p_{\alpha_1}, \ldots, p_{\alpha_{j+k}}], \end{aligned}$$

where $\epsilon(\sigma,\tau)$ *is determined by:*

$$orientation(\sigma) \cdot orientation(\tau) = \epsilon(\sigma,\tau) \cdot orientation(\upsilon)$$

Proof. Recall that for any simplex α, $W\alpha = 0$ on $M \backslash \overline{St(\alpha)}$. So, $\sigma \cup \tau = R(W\sigma \wedge W\tau)$ is zero if their vertices are disjoint. If σ and τ intersect in more than one vertex then $W\sigma \wedge W\tau = 0$ since it is a sum of terms containing $d\mu_{\alpha_i} \wedge d\mu_{\alpha_i}$ for some i. Thus, by possibly reordering the vertices of K, it suffices to show that for $\sigma = [p_0, p_1, \ldots, p_j]$ and $\tau = [p_j, p_{j+1}, \ldots, p_{j+k}]$, we have that $(\sigma \cup \tau)([p_0, p_1, \ldots, p_{j+k}]) = \epsilon(\sigma,\tau)\frac{j!k!}{(j+k+1)!}$. We calculate

$$\begin{aligned} & R(W\sigma \wedge W\tau)([p_0, p_1, \ldots, p_{j+k}]) \\ &= \int_{\upsilon=[p_0,p_1,\ldots,p_{j+k}]} W([p_0, p_1, \ldots, p_j]) \wedge W([p_j, p_{j+1}, \ldots, p_{j+k}]) \\ &= j!k! \int_\upsilon \sum_{i=0}^{j+k} (-1)^i \mu_i \mu_j \; d\mu_0 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_{j+k} \end{aligned}$$

Now, $\sum_{i=0}^{j+k} \mu_i = 1$, so $d\mu_0 = -\sum_{i=0}^{j+k} d\mu_i$, and we have that the last expression

$$\begin{aligned} &= j!k! \int_\upsilon \sum_{i=0}^{j+k} (-1)^i \mu_i \mu_j \; (-d\mu_i) \wedge d\mu_1 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_{j+k} \\ &= j!k! \int_\upsilon \mu_j \sum_{i=0}^{j+k} \mu_i \; d\mu_1 \wedge \cdots \wedge d\mu_{j+k} \\ &= j!k! \int_\upsilon \mu_j \; d\mu_1 \wedge \cdots \wedge d\mu_{j+k} \end{aligned}$$

Now, $|\int_\upsilon d\mu_1 \wedge \cdots \wedge d\mu_{j+k}|$ is the volume of a standard $(j+k)$-simplex, and thus equals $\frac{1}{(j+k)!}$. From this it is easy to show that $\int_\upsilon \mu_j \; d\mu_1 \wedge \cdots \wedge d\mu_{j+k} = \pm\frac{1}{(j+k+1)!}$, with the appropriate sign prescribed by the definition of $\epsilon(\sigma,\tau)$. □

A special case of this result was derived by Ranicki and Sullivan [24] for K a triangulation of a $4k$-manifold and σ, τ of complimentary dimension. In that paper, they showed that the pairing given by $\cup$ restricted to simplices of complimentary dimension gives rise to a semi-local combinatorial formula for the signature of a $4k$-manifold.

Remark 5.3. *The constant 0-cochain which evaluates to 1 on all 0-simplices is the unit of the differential graded commutative (but non-associative) algebra* $(C^*, \delta, \cup)$.

We now show that the product $\cup$ converges to $\wedge$, which perhaps is not surprising, since $\cup$ is induced by the Whitney embedding and the wedge product. Still, the statement may be of computational interest since it shows that in using cochains to approximate differential forms, the product $\cup$ is, in a analytically precise way, an appropriate analogue of the wedge product of forms.

Theorem 5.4. *Let* $\omega_1, \omega_2 \in \Omega(M)$ *and* σ *be an* n*-simplex of* K. *Then there exists a constant* C *independent of* ω_1, ω_2, K *and* σ *such that*

$$|W(R\omega_1 \cup R\omega_2)(p) - \omega_1 \wedge \omega_2(p)|_p \leq C \cdot \left(c_1 \cdot sup\left|\frac{\partial \omega_2}{\partial x^i}\right| + c_2 \cdot sup\left|\frac{\partial \omega_1}{\partial x^i}\right|\right) \cdot \eta$$

for all $p \in \sigma$, *where* $c_m = sup|\omega_m|_p$, *the supremum is over all* $i = 1, 2, \ldots n$, *and the partial derivatives are taken with respect to a coordinate neighborhood containing* σ.

Remark 5.5. *By Remark 5.3, Theorem 5.4 reduces to Theorem 4.5 when* ω_1 *is the constant function 1.*

Proof. Let $\sigma = [p_0, \ldots, p_n]$ be an n-simplex contained in a coordinate neighborhood with coordinate functions $x_1, \ldots, x_n$. Let μ_i denote the i^{th} barycentric coordinate of σ. By the triangle inequality, and a possible reordering of the coordinate functions, it suffices to consider the case

$$\begin{aligned} \omega_1 &= f \; d\mu_1 \wedge \cdots \wedge d\mu_j \\ \omega_2 &= g \; d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k} \end{aligned}$$

We first compute $W(R\omega_1 \cup R\omega_2)$. We'll use the notation $[p_s, \ldots, p_{s+t}]$ to denote both the simplicial chain and the simplicial cochain taking the value one on this chain and zero elsewhere. Let

$$\begin{aligned} N &= \{0, 1, 2, \ldots, n\} \\ J &= \{1, 2, \ldots, j\} \\ K &= \{\alpha_1, \ldots, \alpha_k\}. \end{aligned}$$

Then

$$\begin{aligned} R\omega_1 &= \sum_{\beta \in N-J} \left(\int_{[p_\beta, p_1, \ldots, p_j]} \omega_1 \right) [p_\beta, p_1, \ldots, p_j] \\ R\omega_2 &= \sum_{\gamma \in N-K} \left(\int_{[p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}]} \omega_2 \right) [p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}]. \end{aligned}$$

Now, to compute $R\omega_1 \cup R\omega_2$, we use theorem 5.2. If the sets J and K intersect in two or more elements then $R\omega_1 \cup R\omega_2 = 0$ since, in this case, all products of simplices are zero.

Now suppose that J and K intersect in exactly one element. Without loss of generality, let us assume $\alpha_1 = 1$. Then the product

$$[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}]$$

is non-zero only if β, γ are distinct elements of the set $Q = N - (J \bigcup K)$. Using the abbreviated notation

$$\begin{aligned}
[p_s, p_J, p_K] &= [p_s, p_1, \ldots, p_j, p_{\alpha_1}, \ldots, p_{\alpha_k}] \\
\int_{[s]} \omega_1 &= \int_{[p_s, p_1, \ldots, p_j]} \omega_1 \\
\int_{[s]} \omega_2 &= \int_{[p_s, p_{\alpha_1}, \cdots, p_{\alpha_k}]} \omega_2
\end{aligned}$$

we compute

$$R\omega_1 \cup R\omega_2 = \frac{j!k!}{(j+k+1)!} \sum_{\substack{\beta,\gamma \in Q \\ \beta \neq \gamma}} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) \; [p_\beta, p_\gamma, p_J, p_K].$$

If all of the coefficients (given by the integrals of ω_1 and ω_2) were equal, the above expression would vanish, since the terms would cancel in pairs (by reversing the roles of β and γ). Of course, this is not the case, but the terms are *almost* equal. We'll use some estimation techniques developed by Dodziuk and Patodi [10].

An essential estimate that we'll need for this case and the next is the following: there is a constant c, independent of ω_1, ω_2, K and σ, such that for any $p \in \sigma$, and β, γ as above,

$$(1) \quad \Big| j!k! \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - f(p)g(p) \Big| \; \leq \; c \cdot \Big(c_1 \cdot sup \Big| \frac{\partial \omega_2}{\partial x^j} \Big| + c_2 \cdot sup \Big| \frac{\partial \omega_1}{\partial x^j} \Big| \Big) \cdot \eta^{j+k+1}$$

where $c_m = sup|\omega_m|_p$ and the supremums are taken over all $p \in \sigma$ and $i = 1, 2, \ldots n$.

To prove this, first note that by the mean value theorem, for any points $p, q \in \sigma$, $|\omega_1(q) - \omega_1(p)|_q \leq c \cdot sup|\frac{\partial \omega_1}{\partial x^j}| \cdot \eta$. (Here we're using the fact that the Riemannian metric and the flat one induced by pulling back along the coordinates x^i are commensurable.) Similarly for ω_2. Now, fix $p \in \sigma$ and let dV_β be the volume element on $[p_\beta, p_1, \ldots, p_j]$, and dV_γ be the volume element on $[p_\gamma, p_1, \ldots, p_j]$. Then

$$\begin{aligned}
&\Big| j!k! \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - f(p)g(p) \Big| \\
&= \Big| j!k! \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - \frac{\int_{[\beta]} f(p) d\mu_1 \wedge \cdots \wedge d\mu_j}{\int_{[\beta]} d\mu_1 \wedge \cdots \wedge d\mu_j} \frac{\int_{[\gamma]} g(p) d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k}}{\int_{[\gamma]} d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k}} \Big| \\
&= j!k! \Big| \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - \int_{[\beta]} \omega_1(p) \int_{[\gamma]} \omega_2(p) \Big| \\
&\leq j!k! \Big| \int_{[\beta]} \omega_1 \Big| \Big| \int_{[\gamma]} \omega_2 - \int_{[\gamma]} \omega_2(p) \Big| + \Big| \int_{[\gamma]} \omega_2 \Big| \Big| \int_{[\beta]} \omega_1 - \int_{[\beta]} \omega_1(p) \Big| \\
&\leq j!k! \; c_1 \cdot \eta^j \int_{[\gamma]} |\omega_2 - \omega_2(p)|_q \; dV_\gamma + c_2 \cdot \eta^k \int_{[\beta]} |\omega_1 - \omega_1(p)|_q \; dV_\beta \\
&\leq c \cdot \Big(c_1 \cdot sup \Big| \frac{\partial \omega_2}{\partial x^i} \Big| + c_2 \cdot sup \Big| \frac{\partial \omega_1}{\partial x^i} \Big| \Big) \cdot \eta^{j+k+1}.
\end{aligned}$$

This implies, by the triangle inequality, for any β, γ

$$(2) \quad \Big| \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - \int_{[\gamma]} \omega_1 \int_{[\beta]} \omega_2 \Big| \; \leq \; c \cdot \Big(c_1 \cdot sup \Big| \frac{\partial \omega_2}{\partial x^j} \Big| + c_2 \cdot sup \Big| \frac{\partial \omega_1}{\partial x^j} \Big| \Big) \cdot \eta^{j+k+1}$$

Now that we have estimated the coefficients of $W(R\omega_1 \cup R\omega_2)$, this case is completed by estimating the the product of the $d\mu_i$'s that appear in $W(R\omega_1 \cup \omega_2)$. As shown in [10],

$$|d\mu_i|_p \leq \frac{\lambda}{r(p_i, |\sigma_i|)},$$

where $\sigma_i = [p_0, \cdots, p_{j-1}, p_{j+1}, \cdots, p_N]$ is the face opposite of p_i, and r is the Riemannian geodesic distance. So, by Lemma 4.4

$$|d\mu_i|_p \leq \lambda' \cdot \eta^{-1}$$

for some constant λ', and therefore

$$|d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_{j+k}}|_p \leq |d\mu_{i_1}|_p \ldots |d\mu_{i_{j+k}}|_p \leq \lambda \cdot \eta^{-(j+k)}. \tag{3}$$

By combining (2) and (3), we finally have, for the case that J and K intersect in exactly one element,

$$\begin{aligned} |W(R\omega_1 \cup R\omega_2)(p) - \omega_1 \wedge \omega_2(p)|_p &= |W(R\omega_1 \cup R\omega_2)(p)|_p \\ &\leq C \cdot \left(c_1 \cdot sup\left|\frac{\partial \omega_2}{\partial x^i}\right| + c_2 \cdot sup\left|\frac{\partial \omega_1}{\partial x^i}\right| \right) \cdot \eta \end{aligned}$$

We now consider the case that J and K are disjoint. We first note that for any $\tau \in Q = N - (J \cup K)$, there are exactly $j + k + 1$ products

$$[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}]$$

which equal a nonzero multiple of $[p_\tau, p_J, p_K] = [p_\tau, p_1, \ldots, p_j, p_{\alpha_1}, \ldots, p_{\alpha_k}]$. These are given by the three mutually exclusive cases:

$$\begin{aligned} \beta &= \tau,\ \gamma \in J \\ \gamma &= \tau,\ \beta \in K \\ \beta &= \gamma = \tau \end{aligned}$$

Using the same notation as the previous case, we compute

$$\begin{aligned} R\omega_1 \cup R\omega_2 = {} & \frac{j!k!}{(j+k+1)!}\Bigg(\sum_{\|0\|} \Big(\int_{[\beta]} \omega_1\Big)\Big(\int_{[\gamma]} \omega_2\Big)\ [p_0, p_J, p_K] \\ & + \sum_{\tau \in Q - \{0\}} \sum_{\|\tau\|} \Big(\int_{[\beta]} \omega_1\Big)\Big(\int_{[\gamma]} \omega_2\Big)\ [p_\tau, p_J, p_K]\Bigg) \end{aligned} \tag{4}$$

where the sums labeled $\sum_{\|s\|}$ are over all β, γ such that

$$[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}] = \frac{j!k!}{(j+k+1)!}[p_s, p_J, p_K].$$

From Lemma 5.6, which follows the proof of this theorem,

$$W([p_0, p_J, p_K]) = (j+k)!\ d\mu_J \wedge d\mu_K - \sum_{r \in Q - \{0\}} W([p_\tau, p_J, p_K])$$

So,

$$
\begin{aligned}
&|W(R\omega_1 \cup R\omega_2) - \omega_1 \wedge \omega_2|_p \\
&\leq \frac{j!k!}{(j+k+1)} \left| \sum_{\|0\|} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) \, d\mu_J \wedge d\mu_K - \omega_1 \wedge \omega_2 \right|_p \\
&\quad + \frac{j!k!}{(j+k+1)!} \Bigg| \sum_{\tau \in Q - \{0\}} \Bigg(\sum_{\|\tau\|} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) \\
&\qquad\qquad - \sum_{\|0\|} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) \Bigg) \, W([p_\tau, p_J, p_K]) \Bigg|_p
\end{aligned}
$$

By our estimates in (2) and (3), the latter term is bounded appropriately. As for the first term, recall that the sum $\sum_{\|0\|}$ consists of j+k+1 terms. We use (2) again to bound

$$
\left| \sum_{\|0\|} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) - (j+k+1) \Big(\int_{[0]} \omega_1 \Big) \Big(\int_{[0]} \omega_2 \Big) \right| \tag{5}
$$

and using (1), for fixed $p \in \sigma$ we have a bound on

$$
\left| \Big(\int_{[0]} \omega_1 \Big) \Big(\int_{[0]} \omega_2 \Big) - f(p)g(p) \right|. \tag{6}
$$

Finally, using the triangle inequality and combining (5) and (6) with (3) we can conclude

$$
\begin{aligned}
&\frac{j!k!}{(j+k+1)} \left| \sum_{\|0\|} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) \, d\mu_J \wedge d\mu_K(p) - \omega_1 \wedge \omega_2(p) \right|_p \\
&\qquad\qquad \leq C \cdot \left(c_1 \cdot sup \left| \frac{\partial \omega_2}{\partial x^j} \right| + c_2 \cdot sup \left| \frac{\partial \omega_1}{\partial x^j} \right| \right) \cdot \eta
\end{aligned}
$$

□

Lemma 5.6. *Let* $\sigma = [p_0, p_1, \ldots, p_n]$, $N = \{1, 2, \ldots, n\}$ *and* $I = \{i_1, \ldots i_m\} \subset N$. *Then*

$$
W([p_0, p_{i_1}, \ldots, p_{i_m}]) = m! \; d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_m} - \sum_{r \in N - I} W([p_r, p_{i_1}, \ldots, p_{i_m}])
$$

Proof. The proof is a computation. We let

$$
\begin{aligned}
d\mu_I &= d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_m} \\
d\mu_I^s &= d\mu_{i_1} \wedge \cdots \wedge \widehat{d\mu_{i_s}} \wedge \cdots \wedge d\mu_{i_m}
\end{aligned}
$$

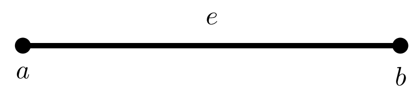

FIGURE 1. Cochain product on the unit interval

and compute

$$\begin{aligned}
\frac{1}{m!}W([p_0, p_{i_1}, \ldots, p_{i_m}]) &= \mu_0 \; d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} \; d\mu_0 \wedge d\mu_I^s \\
&= \Big(1 - \sum_{r=1}^{n} \mu_r\Big) \; d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} \; \Big(-\sum_{r=1}^{n} d\mu_r\Big) \wedge d\mu_I^s \\
&= d\mu_I - \sum_{r=1}^{n} \mu_r \; d\mu_I - \sum_{s=1}^{m} (-1)^s \mu_{i_s} \; \Big(d\mu_{i_s} + \sum_{r \in N-I} d\mu_r\Big) \wedge d\mu_I^s \\
&= d\mu_I - \sum_{r \in N-I} \mu_r \; d\mu_I - \sum_{s=1}^{m} (-1)^s \mu_{i_s} \Big(\sum_{r \in N-I} d\mu_r\Big) \wedge d\mu_I^s \\
&= d\mu_i - \sum_{r \in N-I} \Big(\mu_r \; d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} \; d\mu_r \wedge d\mu_I^s\Big) \\
&= d\mu_I - \frac{1}{m!} \sum_{r \in N-I} W([p_r, p_{i_1}, \ldots, p_{i_m}])
\end{aligned}$$

□

Corollary 5.7. *There exist a constant C and positive integer m, independent of K such that*

$$\|W(R\omega_1 \cup R\omega_2) - \omega_1 \wedge \omega_2\| \leq C \cdot \lambda(\omega_1, \omega_2) \cdot \eta$$

where

$$\lambda(\omega_1, \omega_2) = \|\omega_1\|_\infty \cdot \|(Id + \Delta)^m \omega_2\| + \|\omega_2\|_\infty \cdot \|(Id + \Delta)^m \omega_1\|$$

for all smooth forms $\omega_1, \omega_2 \in \Omega(M)$, where $\| \quad \|$ is the $\mathcal{L}_2$-norm on M.

Proof. We integrate the point-wise estimate from Theorem 5.4, using the fact that M is compact and $sup|\omega_k| = \|\omega_k\|_\infty$, and the Sobolev-Inequality

$$sup\left|\frac{\partial \omega_k}{\partial x^i}\right| \leq C \cdot \|\omega_k\|_{2m} = C \cdot \|(Id + \Delta)^m \omega_k\|$$

for sufficiently large m, where $\| \quad \|_{2m}$ is the Sobolev $2m$-norm. □

The convergence of $\cup$ to the associative product $\wedge$ is, a priori, a bit mysterious due to the following:

Example 5.8. *The product $\cup$ is not associative. For example, in Figure 1, $(a \cup b) \cup e = 0$, since a and b do not span a 0-simplex, but $a \cup (b \cup e) = \frac{1}{4}e$.*

In the above example, the cochains a, b and e may be thought of as "delta functions", in the sense that they evaluate to one on a single simplex and zero elsewhere. If we work with cochains which are "smoother", i.e. represented by the integral of a smooth differential form, associativity is *almost* obtained. In fact, the next theorem shows that for such cochains, the deviation from being associative is bounded by a constant times the mesh of the triangulation. Hence, associativity is recovered in the mesh goes to zero limit.

Theorem 5.9. *There exist a constant C and positive integer m, independent of K such that*

$$\|(R\omega_1 \cup R\omega_2) \cup R\omega_3 - R\omega_1(R\omega_2 \cup R\omega_3)\| \leq C \cdot \lambda(\omega_1, \omega_2, \omega_3) \cdot \eta$$

for all $\omega_1, \omega_2, \omega_3 \in \Omega(M)$, where $\| \quad \|$ is the Whitney norm and

$$\lambda(\omega_1, \omega_2, \omega_3) = \sum \|\omega_r\|_\infty \cdot \|\omega_s\|_\infty \cdot \|(Id + \Delta)^m \omega_t\|$$

where the sum is over all cyclic permutations $\{r, s, t\}$ of $\{1, 2, 3\}$.

Proof. We can prove this by first showing each of $(R\omega_1 \cup R\omega_2) \cup R\omega_3$ and $R\omega_1 \cup (R\omega_2 \cup R\omega_3)$ are close to $\omega_1 \wedge \omega_2 \wedge \omega_3$ in the point-wise norm $| \quad |_p$. The final result is then obtained by integrating and applying the Sobolev inequality to each point-wise error, then applying the triangle inequality.

Let $A \approx B$ mean

$$|A - B|_p \leq c \cdot \sum \|\omega_r\|_\infty \cdot \|\omega_s\|_\infty \cdot sup\left|\frac{\partial \omega_t}{\partial x^i}\right| \cdot \eta$$

We'll only consider the first case

$$W((R\omega_1 \cup R\omega_2) \cup R\omega_3) \approx \omega_1 \wedge \omega_2 \wedge \omega_3; \tag{7}$$

the second case is similar.

It suffices to consider the case

$$\begin{aligned}
\omega_1 &= f \; d\mu_1 \wedge \cdots \wedge d\mu_j \\
\omega_2 &= g \; d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k} \\
\omega_3 &= h \; d\mu_{\beta_1} \wedge \cdots \wedge d\mu_{\beta_l}.
\end{aligned}$$

The proof is analogous to that of Theorem 5.4; the only differences are that the combinatorics of two cochain products is slightly more complicated, and the estimates now involve coefficients which are triple products of integrals over simplices. Let

$$\begin{aligned}
N &= \{1, \ldots, n\} \\
J &= \{1, \ldots, j\} \\
K &= \{\alpha_1, \ldots, \alpha_k\} \\
L &= \{\beta_1, \ldots, \beta_l\} \\
Q &= N - (J \cup K \cup L)
\end{aligned}$$

Let us assume $J \cap K \cap L = \emptyset$; the other cases are similar. Define $A \sim B$ by

$$|A - B| \leq C \cdot \|\omega_r\|_\infty \cdot \|\omega_s\|_\infty \cdot sup\left|\frac{\partial \omega_t}{\partial x^i}\right| \cdot \eta^{j+k+l+1}$$

Using similar techniques as in the proof of theorem 5.4, for all $a \in N-J$, $b \in N-K$, $c \in N-L$

$$\Big(\int_{[a]} \omega_1\Big)\Big(\int_{[b]} \omega_2\Big)\Big(\int_{[c]} \omega_3\Big) \sim \Big(\int_{[0]} \omega_1\Big)\Big(\int_{[0]} \omega_2\Big)\Big(\int_{[0]} \omega_3\Big) \tag{8}$$

$$j!k!l!\Big(\int_{[0]} \omega_1\Big)\Big(\int_{[0]} \omega_2\Big)\Big(\int_{[0]} \omega_3\Big) \sim f(p)g(p)h(p)$$

For any $\tau \in Q$, there are exactly

$$(j+k+1)(j+k+1)+(j+k+1)l = (j+k+1)(j+k+l+1)$$

products

$$[p_a, p_1, \ldots, p_j] \cup [p_b, p_{\alpha_1}, \ldots, p_{\alpha_k}] \cup [p_c, p_{\beta_1}, \ldots, p_{\beta_l}]$$

that equal a non-zero multiple of $[p_\tau, p_J, p_K, p_L]$. Then

$$\frac{j!k!(j+k!)l!}{(j+k+1)!(j+k+l+1)!}(j+k+1)(j+k+l+1) = \frac{j!k!l!}{(j+k+l)!}$$

so that, by applying lemma 5.6, and equations (8) and (3),

$$\begin{aligned} &W((R\omega_1 \cup R\omega_2) \cup R\omega_3) \\ &\approx \frac{j!k!l!}{(j+k+l)!}\Bigg(\Big(\int_{[0]} \omega_1\Big)\Big(\int_{[0]} \omega_2\Big)\Big(\int_{[0]} \omega_3\Big)\, W([p_0, p_J, p_K, p_L]) \\ &\quad + \sum_{\tau \in Q - \{0\}} \Big(\int_{[0]} \omega_1\Big)\Big(\int_{[0]} \omega_2\Big)\Big(\int_{[0]} \omega_3\Big)\, W([p_\tau, p_J, p_K, p_L])\Bigg) \end{aligned}$$

and this is $\approx \omega_1 \wedge \omega_2 \wedge \omega_3$ by (8). □

In the previous theorem, we dealt with the non-associativity of $\cup$ analytically. There is also an algebraic way to deal with this, via an algebraic generalization of commutative, associative algebras, called $\mathcal{C}_\infty$-algebras. First we'll give an abstract definition, and then unravel what it means.

Definition 5.10. *Let C be a graded vector space, and let $C[-1]$ denote the graded vector space C with grading shifted down by one. Let $\mathcal{L}(C) = \bigoplus_i \mathcal{L}^i(C)$ be the free Lie co-algebra on C. A $\mathcal{C}_\infty$-algebra structure on C is a degree 1 co-derivation $D : \mathcal{L}(C[-1]) \to \mathcal{L}(C[-1])$ such that $D^2 = 0$.*

A co-derivation on a free Lie co-algebra is uniquely determined by a collection of maps from $\mathcal{L}^i(C)$ to C for each $i \geq 1$. If we let m_i denote the restriction of D to $\mathcal{L}^i(C)$, then the equation $D^2 = 0$ unravels to a collection of equations:

$$\begin{aligned} m_1^2 &= 0 \\ m_1 \circ m_2 &= m_2 \circ m_1 \\ m_2 \circ m_2 - m_2 \circ m_2 &= m_1 \circ m_3 + m_3 \circ m_1 \\ &\vdots \end{aligned}$$

We can regard m_1 as a differential and m_2 a commutative multiplication on C. The second equation states that m_1 is a derivation of m_2. The third equation states that m_2 is associative up to the (co)-chain homotopy m_3. Note that, due to the shift of grading, m_j has degree $2-j$.

The following theorem is due to Sullivan, see [32] for an exposition and use of similar techniques.

Theorem 5.11. *Let (C, δ) be the simplicial cochains of a triangulated space and $\cup$ be any local commutative (possibly non-associative) cochain multiplication on C such that δ is a derivation of $\cup$. Then there is a canonical local inductive construction which extends $(C, \delta, \cup)$ to a $\mathcal{C}_\infty$-algebra.*

In this theorem, local means that the product of a j-simplex and a k-simplex is zero unless they span a $j+k$-simplex, in which case it is a multiple of this simplex. By theorem 5.2, the commutative product $\cup$ defined at the beginning of this section satisfies this and the other conditions of theorem 5.11.

The next theorem shows that the $\mathcal{C}_\infty$-algebra on C converges to the strict commutative and associative algebra given by the wedge product on forms in a sense analogous to the convergence statements we've made previously. In particular, all higher homotopies converge to zero as the mesh tends to zero.

Theorem 5.12. *Let C be the simplicial cochains of a triangulation K of M, with mesh $0 \le \eta \le 1$. Let $m_1 = \delta, m_2 = \cup, m_3, \ldots$ be the extension of $\delta, \cup$ to a $\mathcal{C}_\infty$-algebra on C as in theorem 5.11. Then there exists a constant λ independent of K such that, for all $j \ge 3$,*

$$\|W(m_j(R\omega_1, \ldots, R\omega_j))\| \le \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_\infty \cdot \eta$$

for all $\omega_1, \ldots, \omega_k \in \Omega(M)$.

Proof. Suppose $\omega_1, \ldots, \omega_j$ are of degree $\alpha_1, \ldots, \alpha_j$, respectively. Let $\alpha = \sum \alpha_i$. We need two facts. First, for any α_i-simplex τ of K,

$$|R\omega_i(\tau)| \le c \cdot \|\omega_i\|_\infty \cdot \eta^{\alpha_i} \tag{9}$$

Secondly, if p is a point in an n-simplex σ, and the r-simplices which are faces of σ are $\sigma_r^1, \ldots, \sigma_r^m$ then, by equation (3),

$$\left| W\left(\sum_{i=1}^{m} \sigma_r^i \right) \right|_p \le c' \cdot \eta^{-r}. \tag{10}$$

Now, since m_j has degree $2-j$, $m_j(R\omega_1, \ldots, R\omega_j)$ is a linear combination of $(\alpha + 2 - j)$-simplices. Combining this with (9) and (10), we have for all $p \in M$ and some $\lambda \ge 0$

$$\begin{aligned} |W(m_j(R\omega_1, \ldots, R\omega_j))|_p &\le \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_\infty \cdot \eta^{\alpha} \cdot \eta^{-(\alpha+2-j)} \\ &\le \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_\infty \cdot \eta \end{aligned}$$

The result is obtained by integrating over M. □

6. Combinatorial Star Operator

In this section we define the combinatorial star operator $\bigstar$ and prove that it provides a good approximation to the smooth Hodge-star $\star$. We also examine the relations which are expected to hold by analogy with the smooth setting. We find that some hold precisely, while others may only be recovered as the mesh goes to zero.

Definition 6.1. *Let K be a triangulation of a closed orientable manifold M, with simplicial cochains $C = \bigoplus_j C^j$. Let $\langle , \rangle$ be a non-degenerate positive definite inner product on C such that C^i is orthogonal to C^j for $i \neq j$. For $\sigma \in C^j$ we define $\bigstar\sigma \in C^{n-j}$ by:*

$$\langle \bigstar\sigma, \tau\rangle = (\sigma \cup \tau)[M]$$

where $[M]$ denotes the fundamental class of M.

We emphasize that, as exemplified by definition 3.2, the essential ingredients of a star operator are Poincaré Duality and a non-degenerate inner product. We can regard the inner product as giving some geometric structure to the space. In particular it gives lengths of edges, and angles between them. As in the smooth setting, the star operator depends on the choice of inner product (or Riemannian metric). See section 7 for the definition of a particularly nice class of inner products that we call *geometric inner products*.

Here are some elementary properties of $\bigstar$.

Lemma 6.2. *The following hold:*

1. $\bigstar\delta = (-1)^{j+1}\delta^*\bigstar$*, i.e.* $\bigstar$ *is a chain map.*
2. *For $\sigma \in C^j$ and $\tau \in C^{n-j}$, $\langle \bigstar\sigma, \tau\rangle = (-1)^{j(n-j)}\langle \sigma, \bigstar\tau\rangle$, i.e. $\bigstar$ is (graded) skew-adjoint.*
3. $\bigstar$ *induces isomorphisms $\mathcal{H}C^j(K) \to \mathcal{H}C^{n-j}(K)$ on harmonic cochains.*

Proof. The first two proofs are computational:

(1) For $\sigma, \tau \in C$, we have:

$$\begin{aligned}\langle \bigstar\delta\sigma, \tau\rangle &= (\delta\sigma \cup \tau)[M] \\ &= (-1)^{j+1}(\sigma \cup \delta\tau)[M] \\ &= (-1)^{j+1}\langle \bigstar\sigma, \delta\tau\rangle \\ &= \langle (-1)^{j+1}\delta^*\bigstar\sigma, \tau\rangle\end{aligned}$$

where we have used that fact that d is a derivation of $\cup$ and M is closed.

(2) We compute:

$$\begin{aligned}\langle \bigstar\sigma, \tau\rangle &= (\sigma \cup \tau)[M] \\ &= (-1)^{j(n-j)}(\tau \cup \sigma)[M] \\ &= (-1)^{j(n-j)}\langle \bigstar\tau, \sigma\rangle \\ &= (-1)^{j(n-j)}\langle \sigma, \bigstar\tau\rangle\end{aligned}$$

(3) By the first part of the lemma, $\bigstar$ maps $Ker(\delta)$ to $Ker(\delta^*)$ and $Im(\delta)$ to $Im(\delta^*)$. By the Hodge decomposition, the quotients $Ker(\delta)/Im(\delta)$ and $Ker(\delta^*)/Im(\delta^*)$ can be identified with the harmonic cochains, and the map $\bigstar$ on the quotients can identify as the composition of Poincaré Duality and the inverse of the non-degenerate metric. The former induces an ismorphism by the Poincaré Duality theorem for manifolds, and the latter is an ismorphism by non-degeneracy.

□

We remark here that $\bigstar$ is in general not invertible, since the cochain product does not necessarily give rise to a non-degenerate pairing (on the cochain level!). This implies that $\bigstar$ is not an orthogonal map, and $\bigstar^2 \neq \pm\mathrm{Id}$.

For the remainder of this section, we'll fix the inner product on cochains to be the Whitney inner product, so that $\bigstar$ is the star operator induced by the Whitney inner product. This will be essential in showing that $\bigstar$ converges to the smooth Hodge star $\star$, which is defined using the Riemannian metric. First, a useful lemma. Let π denote the orthogonal projection of $\Omega^j(M)$ onto the image of $C^j(K)$ under the Whitney embedding W.

Lemma 6.3. $W\bigstar = \pi \star W$

Proof. Let $a \in C^j(K)$ and $b \in C^{n-j}(K)$. Note that $\star Wa$ is an $\mathcal{L}_2$-form but in general is not a Whitney form. We compute:

$$\langle W\bigstar a, Wb\rangle = \langle \bigstar a, b\rangle = \int_M Wa \wedge Wb = \langle \star Wa, Wb\rangle,$$

Thus, $W\bigstar a$ and $\star Wa$ have the same inner product with all forms in the image of W, so $W\bigstar = \pi \star W$. □

Now for our convergence theorem of $\bigstar$:

Theorem 6.4. *Let M be a Riemannian manifold with triangulation K of mesh η. There exist a positive constant C and a positive integer m, independent of K, such that*

$$\|\star\omega - W\bigstar R\omega\| \leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta$$

for all C^∞ differential forms ω on M.

Proof. We compute and use Corollary 4.6

$$\begin{aligned}
\|\star\omega - W\bigstar R\omega\| &= \|\star\omega - \pi \star WR\omega\| \\
&\leq \|\star\omega - \star WR\omega\| + \|\star WR\omega - \pi \star WR\omega\| \\
&\leq \|\star\|\|\omega - WR\omega\| + \|\star WR\omega - WR\star\omega\| \\
&\leq \|\omega - WR\omega\| + \|\star WR\omega - \star\omega\| + \|\star\omega - WR\star\omega\| \\
&\leq 2\|\omega - WR\omega\| + \|\star\omega - WR\star\omega\| \\
&\leq 3C \cdot \|(Id + \Delta)^m\omega\| \cdot \eta
\end{aligned}$$

□

The operator $\bigstar$ also respects the Hodge decompositions of $C(K)$ and $\Omega(M)$ in the following sense:

Theorem 6.5. *Let M be a Riemannian manifold with triangulation K of mesh η. Let $\omega \in \Omega^j(M)$ and $R\omega \in C^j(K)$ have Hodge decompositions*

$$\begin{aligned}
\omega &= d\omega_1 + \omega_2 + d^*\omega_3 \\
R\omega &= \delta a_1 + a_2 + \delta^* a_3
\end{aligned}$$

There exist a positive constant C and a positive integer m, independent of K, such that

$$\begin{aligned}
\| \star d\omega_1 - W\bigstar\delta a_1\| &\leq C \cdot (\|(Id + \Delta)^m\omega\| + \|(Id + \Delta)^m d\omega_1\|) \cdot \eta \\
\| \star \omega_2 - W\bigstar a_2\| &\leq C \cdot (\|(Id + \Delta)^m\omega\| + \|(Id + \Delta)^m \omega_2\|) \cdot \eta \\
\| \star d^*\omega_3 - W\bigstar\delta^* a_3\| &\leq C \cdot (\|(Id + \Delta)^m\omega\| + \|(Id + \Delta)^m d^*\omega_3\|) \cdot \eta
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 6.4, though for each of the three inequalities we'll use the corresponding estimates in Theorem 4.10. For the first statement we have

$$\begin{aligned}\|\star d\omega_1 - W\bigstar\delta a_1\| &= \|\star d\omega_1 - \pi\star W\delta a_1\| \\ &\leq \|\star d\omega_1 - \star W\delta a_1\| + \|\star W\delta a_1 - \pi\star W\delta a_1\| \\ &\leq \|\star d\omega_1 - \star W\delta a_1\| + \|\star W\delta a_1 - WR\star d\omega_1\| \\ &\leq \|d\omega_1 - W\delta a_1\| + \|\star W\delta a_1 - \star d\omega_1\| + \|\star d\omega_1 - WR\star d\omega_1\|\end{aligned}$$

The first two terms are bound using the first estimate in Theorem 4.10 and the last term is bound using Corollary 4.6 applied to $\star d\omega_1$. The computations proving the last two inequalities in the theorem are the same, with the bounds achieved using the latter two inequalities in Theorem 4.10, and Corollary 4.6 applied to ω_2 and $\star d^*\omega_3$, respectively. □

This section ends with a discussion of convergence for compositions of the operators δ, δ^*, and $\bigstar$. We first note that δ provides a good approximation of d in the sense that $\|d\omega - W\delta R\omega\|$ is bounded by a constant times the mesh. This follows immediately from Theorem 4.5, using the fact that $\delta R = Rd$. In the same way, using Theorem 6.4, $\bigstar\delta$ provides a good approximation to $\star d$. In summary, we have:

$$\pm\delta^*\bigstar = \bigstar\delta \to \star d = \pm d^*\star$$

One would also like to know if either of $\delta\bigstar$ or $\bigstar\delta^*$ provide a good approximation to $d\star$ or $\star d^*$, respectively. Answers to these questions are seemingly harder to come by.

As a precursor, we point out that there is not a complete answer as to whether or not δ^* converges to d^*. In [26], Smits does prove convergence for the case of 1-cochains on a surface. To the author's mind, and as can be seen in the work of [26], one difficulty (with the general case) is that the operator δ^* is not local, since it involves the inverse of the cochain inner product.[1] A first attempt to understand this inverse is described in Section 7.

The issue becomes further complicated when considering the operator $\bigstar\delta^*$. We have no convergence statements about this operator. On the other hand, the operator $\delta\bigstar$, which incidentally does not equal $\pm\bigstar\delta^*$, is a bit less mysterious, and we have weak convergence in the sense that

$$\langle W\delta\bigstar R\omega_1 - d\star\omega_1, \omega_2\rangle$$

is bounded by a constant λ (depending on ω_1 and ω_2) times the mesh.

Finally, one might ask if $\bigstar^2$ approaches $\pm\mathrm{Id}$ for a fine triangulation. While we have no analytic result to state, our calculations for the circle in section 8 suggest this is the case. One can show that a graded symmetric operator squares to $\pm\mathrm{Id}$ if and only if it is orthogonal. Hence one might view $\bigstar^2 \neq \mathrm{Id}$ as the failure of orthogonality, though one can always take the orthogonal part in the polarization.

[1] If the cochain inner product is written as a matrix M with respect to the basis given by the simplices, then $\delta^* = M^{-1}\partial M$ where ∂ is the usual boundary operator on chains.

7. Inner Products and Their Inverses

In this section we study inner products on cochains, as well as the induced "inverse inner product". Smits also studied the inverse of inner products in [26], where he proved results on the convergence of the divergence operator d^* on a surface.

Definition 7.1. *A* geometric inner product *on the real vector space of simplicial cochains $C = \bigoplus_j C^j$ of a triangulated space K is a non-degenerate positive definite inner product $\langle , \rangle$ on C satisfying:*

1. *$C^i \perp C^j$ for $i \neq j$*
2. *locality: $\langle a, b \rangle \neq 0$ only if $St(a) \cap St(b)$ is non-empty.*

Remark 7.2. *A geometric inner product restricted to 1-cochains gives a notion of lengths of edges and the angles between them. It may be interesting to study the consequences of an inner product with signature other than the one considered here.*

We assume in this section that all cochain inner products are geometric in the above sense. Note that the Whitney inner product is geometric.

An inner product on C^* induces an isomorphism from C^* to the linear dual of C^*, which we denote by C_* and refer to as the simplicial chains (to be more precise, this is the double dual of chains, but we'll confuse the two since we're assuming K is compact). The "inverse of the inner product" is that induced by the inverse of the isomorphism $C^* \to C_*$, which is an isomorphism $C_* \to C^*$. This gives an inner product on the (simplicial) chains C_* and will be denoted by $\langle , \rangle^{-1}$.

If one represents a geometric cochain inner product as a matrix, using the standard basis given by the simplices, then the locality property roughly states that this matrix is "near diagonal". Of course, the inverse of a diagonal matrix is diagonal, but the inverse of a near diagonal matrix is *not* near diagonal (e.g. see Section 8). Rather, it can have all entries non-zero; i.e. the inverse inner product on chains is *not* geometric.[2]

In the rest of this section, we describe the inverse inner product $\langle , \rangle^{-1}$ on chains in a geometric way by showing it can be computed as a weighted sum of paths in a collection of graphs associated to K. This will be useful in the next section for making explicit computations of the combinatorial star operator. We begin with some definitions:

Definition 7.3. *A graph Γ (without loops) consists of a set S, called vertices, and a collection of two-element subsets of S, called* edges *Two edges of Γ are said to be incident if their intersection (as subsets of S) is nonempty. A weighted graph is a graph with an assignment of a real number $w(e)$ to each edge e. A path γ in a graph is a sequences of edges $\{e_i\}_{i \in I}$ such that e_i and e_{i+1} are incident for each i. The weight $w(\gamma)$ of a path γ in a weighted graph is the product of the weights of the edges in γ. By convention, we say there is a unique path of length zero between any vertex and itself, and the weight of this path is one.*

Definition 7.4. *Let K be the simplicial cochain complex of a triangulated n-manifold M. We define the* graph associated to the j-simplicies of K, *denoted $\Gamma(K, j)$, to be the following graph: The vertices of $\Gamma(K, j)$ are the set $\{\sigma_\alpha\}$ of*

[2] It is true is that the matrix entries decrease in absolute value as they move from the diagonal, so that the inner product of two chains decays rapidly as a function of "geometric distance".

j-simplices of K; two distinct vertices σ_1, σ_2 of $\Gamma(K, j)$ are joined by an edge if and only if they are faces of a common n-simplex of K (i.e $St(\sigma_1) \cap St(\sigma_2)$ is non-empty).

Corollary 7.5. *Let K be the simplicial cochain complex of a triangulated n-manifold M.*

1. *Paths in $\Gamma(K, j)$ correspond to sequences $\{s_i\}_{i \in I}$ of j-simplices in K such that, for each i, s_i and s_{i+1} are faces of a common n-simplex.*
2. *$\Gamma(K, 0)$ is isomorphic to K_1, the 1-skeleton of K (the union of its vertices and edges).*

Now suppose the cochains C^* of K are endowed with a geometric inner product $\langle , \rangle$. (Our motivating example is the Whitney metric on C^*, but other examples arise when considering interactions on simplicial lattices.) In this case we associate to $(C^*, \langle , \rangle)$ the following collection of weighted graphs.

Definition 7.6. *Let C^* be the cochains of a finite triangulation K of a manifold, with geometric cochain inner product $\langle , \rangle$. We define the* weighted j-cochain graph of K *$\Gamma_w(K, j)$ to have the underlying graph of $\Gamma(K, j)$ with the edge $e = \{\sigma_1, \sigma_2\}$ weighted by*

$$w(e) = \frac{\langle \sigma_1, \sigma_2 \rangle}{\|\sigma_1\| \cdot \|\sigma_2\|}$$

where $\|\sigma\| = \sqrt{\langle \sigma, \sigma \rangle}$

Remark 7.7. *The appropriate analogue of corollary 7.5 for weighted graphs holds as well.*

The following describes how the metric $\langle , \rangle^{-1}$ on C_j can be computed by counting weighted paths in the weighted j-cochain graph $\Gamma_w(K, j)$.

Theorem 7.8. *For $\sigma_1, \sigma_2 \in C_j$*

$$\langle \sigma_1, \sigma_2 \rangle^{-1} = \frac{1}{\|\sigma_1\| \cdot \|\sigma_2\|} \sum_{i \geq 0} (-1)^i \sum_{\gamma_i \in \Gamma_w(K,j)} w(\gamma_i)$$

where γ_i is a path in $\Gamma_w(K, j)$ of length i, starting at σ_1 and ending at σ_2.

Proof. Let M be the matrix for $\langle , \rangle$ with respect to a fixed ordering of the basis given by the simplices of K. Let D be the diagonal matrix, with respect to the same ordered basis, whose diagonal entries are the norm of a simplex. Let $M_D = D^{-1} M D^{-1}$. Note that the entries of M_D are normalized since the entries of D^{-1} are of the form $\frac{1}{\|\sigma\|}$. In particular the diagonal entries of M_D equal 1, so we may write

$$M^{-1} = D^{-1} (M_D)^{-1} D^{-1} = D^{-1} (I + A)^{-1} D^{-1}$$

It is easy to check that A is precisely the weighted adjacency matrix for the weighted graph $\Gamma_w(K, j)$. Recall that the i^{th} power of a weighted adjacency matrix counts the sum of the weights of all paths of length i. By the Cauchy-Schwartz inequality, all of the entries a of A satisfy $0 \leq a < 1$, so the formula

$$(I + A)^{-1} = \sum_{i \geq 0} (-1)^i A^i$$

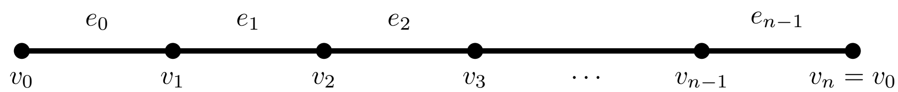

FIGURE 2. Triangulation of S^1

may be applied above, and we conclude that

$$\langle \sigma_1, \sigma_2 \rangle^{-1} = \sigma_1 M^{-1} \sigma_2 = \frac{1}{\|\sigma_1\| \cdot \|\sigma_2\|} \sum_{i \geq 0} (-1)^i \sum_{\gamma_i \in \Gamma_w(K,j)} w(\gamma_i)$$

$\square$

Remark 7.9. (1) *The above theorem in the case $j = 0$, in light of Remark 7.7, shows that for vertices p and q of K, $\langle p, q \rangle^{-1}$ may be expressed as a weighted sum over all paths in the 1-skeleton $K_1 \subset K$.*

(2) *These expressions for $\langle , \rangle^{-1}$ not only provide a nice geometric interpretation, but are also useful for computations, as we will see in Section 8 where we compute $\bigstar$ for the circle.*

8. COMPUTATION FOR S^1

In this section we compute the operator $\bigstar$ explicitly for the circle S^1. We take S^1 to be the unit interval $[0, 1]$ with 0 and 1 identified. We consider a sequence of subdivisions, the n^{th} triangulation being given by vertices at the points $v_i = \frac{i}{n}$ for $0 \leq i \leq n$. We denote the edge from v_i to v_{i+1} by e_i for $0 \leq i \leq n$ and orient this edge from v_i to v_{i+1}. See Figure 2.

All operators will be written as matrices with respect to the ordered basis $\{v_0, \ldots, v_{n-1}, e_0, \ldots, e_{n-1}\}$.

Recall that the operator $\bigstar$ is defined by $\langle \bigstar\sigma, \tau \rangle = (\sigma \cup \tau)[S^1]$ where here $[S^1]$ is the sum of all the edges with their chosen orientations. We'll use the cochain inner product $\langle , \rangle$ induced by the Whitney embedding and the standard metric on S^1 (i.e. $\langle \mathrm{d}t, \mathrm{d}t \rangle = 1$). Let M denote the matrix for the cochain inner product and let C denote the matrix for the pairing given by $(\sigma, \tau) \mapsto (\sigma \cup \tau)[S^1]$. Then $\bigstar = M^{-1}C$. (We suppress the dependence of these operators on the level of subdivision; the n^{th} level M and C are size $2n \times 2n$.)

By the definition of $\cup$ and our chosen orientations we have that

$$C = \left(\begin{array}{c|c} 0 & A \\ \hline A^t & 0 \end{array} \right)$$

where

$$A = \begin{pmatrix} 1/2 & 0 & 0 & \ldots & 0 & 1/2 \\ 1/2 & 1/2 & 0 & \ldots & \ldots & 0 \\ 0 & 1/2 & 1/2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1/2 & 1/2 \end{pmatrix}$$

and t denotes transpose.

One can compute explicitly:

$$\langle \sigma, \tau \rangle = \begin{cases} \frac{2}{3n} & \sigma = \tau \text{ is a vertex} \\ \frac{1}{6n} & \sigma, \tau \text{ are vertices in the boundary of a common edge} \\ n & \sigma = \tau \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

So, in our chosen basis, the matrix for the inner product is given by:

$$M = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & nI \end{array} \right)$$

where I denotes the $n \times n$ identity matrix and

$$B = \begin{pmatrix} 2/3n & 1/6n & 0 & \dots & 0 & 1/6n \\ 1/6n & 2/3n & 1/6n & \dots & \dots & 0 \\ 0 & 1/6n & 2/3n & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \ddots & 1/6n \\ 1/6n & 0 & \dots & 0 & 1/6n & 2/3n \end{pmatrix}.$$

We now compute B^{-1}. Note that one can write $B = \frac{2}{3n}(\frac{1}{4}D + I)$ where

$$D = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

then

$$\begin{aligned} B^{-1} &= \frac{3n}{2}\left(\frac{1}{4}D + I\right)^{-1} \\ &= \frac{3n}{2}(I - \frac{1}{4}D + \frac{1}{4^2}D^2 - \frac{1}{4^3}D^3 \pm \cdots) \\ &= \frac{3n}{2}\sum_{k \geq 0}(-1/4)^k D^k \end{aligned}$$

Note that D is the adjacency matrix for the graph corresponding to the original triangulation K, or rather, $\frac{1}{4}D$ is the weighted adjacency matrix for the weighted graph in Figure 3.

As shown in Section 7, the matrices $\frac{1}{4^k}D^k$ have a geometric interpretation: the (i, j) entry equals the total weight of all paths from v_i to v_j of length k. Since in this case all weights are $\frac{1}{4}$, we'll simply compute the the (i, j) entry of D^k, i.e the total number of paths from v_i to v_j of length k.

We first note that for the real line with integer vertices, the number of paths of length r between two vertices distance s apart is the binomial coefficient $\binom{r}{\frac{r+s}{2}}$. By

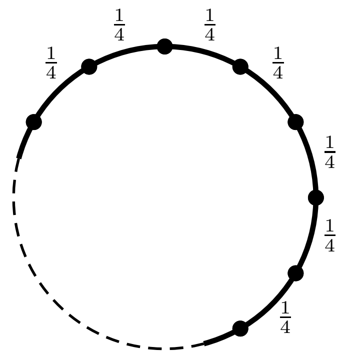

FIGURE 3. The weighted graph corresponding to $\frac{1}{4}D$

considering the standard covering of the circle with n vertices by the line we have

$$d^k_{i,j} = \sum_{t\in\mathbb{Z}} \binom{k}{\frac{k+|i-j|+nt}{2}}$$

where the above binomial coefficient is zero unless $\frac{k+|i-j|+nt}{2}$ is a non-negative integer less than or equal to k. Hence,

$$M^{-1} = \left(\begin{array}{c|c} \left[\frac{3n}{2}\sum_{k\geq 0}\left(\frac{-1}{4}\right)^k d^k_{i,j}\right] & 0 \\ \hline 0 & \frac{1}{n}I \end{array}\right).$$

We conclude that:

$$\begin{aligned} \bigstar v_i &= \frac{1}{2n}(e_{i-1}+e_i) \\ \bigstar e_i &= \frac{3n}{4}\sum_{0\leq j\leq n-1}\left(\sum_{k\geq 0}\left(\frac{-1}{4}\right)^k \sum_{t\in\mathbb{Z}}\binom{k}{\frac{k+|i-j|+nt}{2}} + \binom{k}{\frac{k+|i-(j+1)|+nt}{2}}\right)v_j \end{aligned}$$

In the Figures 4, 5 and 6, we plot $\bigstar e_{n/2}$ for $n = 10, 20, 50$. In each figure, the x-axis denotes the circle, triangulated with black dots as vertices. For fixed n, and each $0 \leq i \leq n$, we plot the coefficient of v_i appearing in $\bigstar e_{n/2}$. We've used a triangle to denote this value. To suggest that the plots are roughly a "delta function" supported in a small neighborhood, we have connected consecutive plot points with a line.

The matrices and plots we have encountered are reminiscent of those that appear in the study of discrete differential operators, and appear to exhibit a sort of "Gibbs phenomenon". We emphasize that this phenomenon results from the structure of the inverse of the cochain inner product. From our computation of $\bigstar$ one can easily compute $\bigstar^2$, and it is clear that this operator approximates a delta-type function.

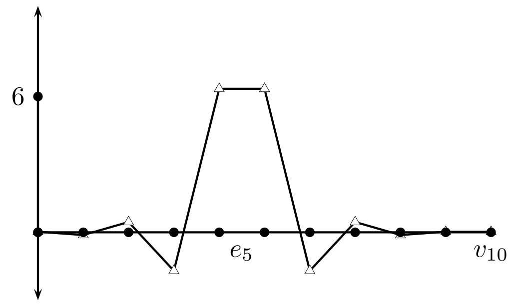

FIGURE 4. Coefficients of v_i appearing in $\bigstar e_5$, for $n = 10$

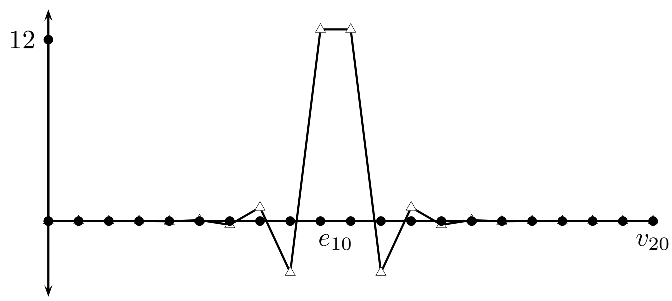

FIGURE 5. Coefficients of v_i appearing in $\bigstar e_{10}$, for $n = 20$

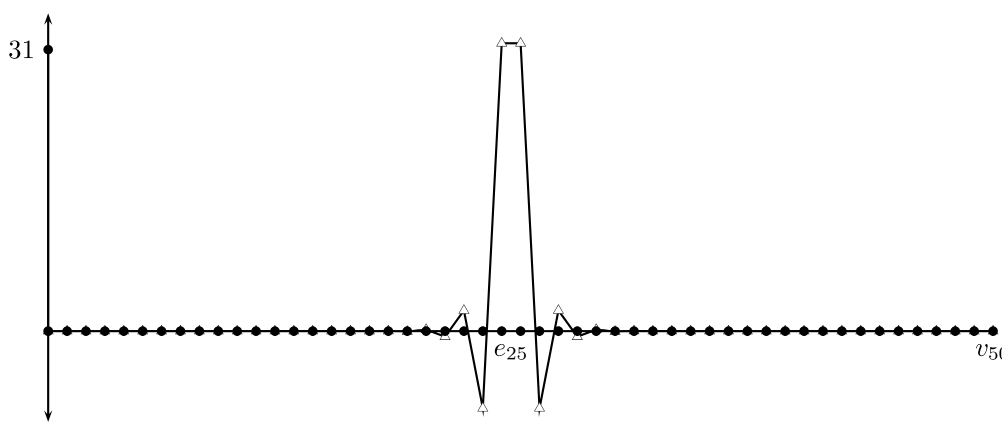

FIGURE 6. Coefficients of v_i appearing in $\bigstar e_{25}$, for $n = 50$

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Scott O. Wilson, 127 Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455.
email: scottw@umn.edu