# Functions

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**Vector functions**

$\vec{\mathbf{r}} : \mathbb{R} \to \mathbb{R}^3$ (or $\mathbb{R}^2$ or $\mathbb{R}^n$)

$\vec{\mathbf{r}} : t \mapsto \langle f(t), g(t), h(t) \rangle$

$\vec{\mathbf{r}}$ takes in a real number $t$

outputs a vector $\langle f(t), g(t), h(t) \rangle$

# Limities and Helices

The **limit** of a vector function $\vec{\mathbf{r}}$ is defined by taking the limits of its component functions

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

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Plug in some values of $t$

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So $\vec{\mathbf{r}} =$

# Derivatives and Derivative-derivative definitions

Define $\vec{\mathbf{r}}'(t) = \lim\limits_{h \to 0} \frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)}{h} = \left\langle f'(t), g'(t), h'(t) \right\rangle$.

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Example. $\int \langle 2\cos t, \sin t, 2t\rangle\,dt = \langle 2\sin t, -\cos t, t^2\rangle + \vec{\mathbf{C}}$.

# Derivatives

Example. Find the equation of the tangent line to the helix $\vec{\mathbf{r}}(t) = \langle 2\cos t, \sin t, t \rangle$ at the point $P = (0, 1, \frac{\pi}{2})$.

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**Game plan:**

1. Find $t^*$ for which the curve goes through the point $P$.

2. Find the tangent vector $\vec{\mathbf{r}}'(t)$, plug in $t = t^*$.

3. Write the equation of the line.

# Derivatives rule

- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t)+\vec{\mathbf{s}}(t)\big)=\vec{\mathbf{r}}'(t)+\vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(c\,\vec{\mathbf{r}}(t)\big)=c\vec{\mathbf{r}}'(t)$
- $\frac{d}{dt}\big(f(t)\,\vec{\mathbf{r}}(t)\big)=f'(t)\,\vec{\mathbf{r}}(t)+f(t)\,\vec{\mathbf{r}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{s}}(t)\big)=\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{s}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t)\times\vec{\mathbf{s}}(t)\big)=\vec{\mathbf{r}}'(t)\times\vec{\mathbf{s}}(t)+\vec{\mathbf{r}}(t)\times\vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(f(t))\big)=f'(t)\,\vec{\mathbf{r}}'(f(t))$