Functions of Several Variables

Function of one variable	**Function of several variables**
$f : \mathbb{R} \to \mathbb{R}$	$f : \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{R}^n \to \mathbb{R}$)
$f : x \mapsto f(x)$	$f : (x, y) \mapsto f(x, y) = z$
f takes in a real number x	f takes in two real numbers x & y
outputs real number $y = f(x)$	outputs real number $z = f(x, y)$
Domain: x-vals where f defined.	Domain: (x, y)-vals where f defined.
Range: y-vals that f can output.	Range: z-vals that f can output.

Three ways to understand functions of two variables:

- What is the domain of the function? (A set in 2D)
- Sketching the graph of a function. (A surface over this set)
- Drawing the level curves of the function (A set of 2D curves)
  - A curve represents points in the domain at the same "height".

The domain of a function of two variables

Example. What is the domain of the functions

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1} \qquad \text{and} \qquad g(x, y) = x \ln(y^2 - x)?$$

Example. Sketch the following functions

$$f(x, y) = 6 - 3x - 2y \qquad \text{and} \qquad g(x, y) = \sqrt{9 - x^2 - y^2}.$$

Level curves

The **level curves** or **contour curves** of a function f are the set of curves of the equations $f(x, y) = k$ for varying constants k.

1. Temperature maps (isothermals)
2. Mountain range maps (contour line)

- Visualize level curves being lifted to piece together the surface.
- Where the lines are close together, the surface is steeper.

Example. What surface corresponds to this contour map?

Example. Sketch the level curves of the function $h(x, y) = \sqrt{9 - x^2 - y^2}$ for $k = 0, 1, 2, 3$.

More variables

We can define functions of more variables $f(x_1, x_2, \ldots, x_n)$.

- Input: n numbers. Output: one number
- The simplest type of function is a linear function: $f(\vec{\mathbf{x}}) = \vec{\mathbf{c}} \cdot \vec{\mathbf{x}}$.

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

  Think: c_i is the unit cost of object i. Cost of x_i units is ___.
  So $f(\vec{\mathbf{x}})$ is the total cost of all objects. (How much was lunch?)
- Can not visualize the surface when $n \geq 3$.
- When $n = 3$, we can understand the **level surfaces** of f.
  (Where is $f(x, y, z) = k$?)

  This gives a *surface* on which the function has a constant value.

Think: Which positions in this room have the same temperature?