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No matter how close to $y = L$ you insist you must be (ε-close),

There is a way to choose a range δ around $x = a$ to ensure that

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Visually:	*Visually:*
Interpretation:	*Interpretation:*
However you approach $x = a$, the value $f(x)$ **always** approaches L.	**However** you approach $(x, y) = (a, b)$, the value $f(x, y)$ **always** approaches L.
Mathematically:	*Mathematically:*
No matter how close to $y = L$ you insist you must be (ε-close),	No matter how close to $z = L$ you insist you must be (ε-close),
There is a way to choose a range δ around $x = a$ to ensure that	There is a way to choose a radius δ around $(x, y) = (a, b)$ to ensure that
All values within δ of a give function values within ε of L.	All values within δ of (a, b) give function values within ε of L.

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The limits along different paths do not agree, so the limit DNE.

More lines of thought

Example. Does the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

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Consequence: If we know $f(x, y)$ is continuous at (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y)$ exists!

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This is the **partial derivative of f with respect to x**. We write:

$$f_x(x, y) \quad \textbf{or} \quad \frac{\partial f}{\partial x} \quad \textbf{or} \quad \frac{\partial}{\partial x} f(x, y) \quad \textbf{or} \quad \frac{\partial z}{\partial x} \quad \textbf{or} \quad D_x f.$$

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★ **Idea:** Treat other variables as constants, differentiate normally. ★

Example. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(2, 1)$ and $f_y(2, 1)$.

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$$\frac{\partial z}{\partial x} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy}$$