Chain Rule

Function of one variable

Suppose $y = f(x)$ and $x = g(t)$.
That is, $y = f\big(g(t)\big)$.
The chain rule gives:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = f'(g(t)) \cdot g'(t)$$

Key idea:
You must add contributions from all dependencies.

Function of several variables

Suppose $z = f(x, y)$ and $\begin{cases} x = g(t) \\ y = h(t) \end{cases}$
So $z = f\big(g(t), h(t)\big)$.

The chain rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = f_x\big(g(t), h(t)\big) \cdot g'(t) + f_y\big(g(t), h(t)\big) \cdot h'(t)$$

Example. Let $z = x^2y + 3xy^2$, where $x = \sin 2t$, $y = \cos t$. Find $\frac{dz}{dt}$, $z'(0)$.
Answer:

More Chains

All dependencies

Alternatively, we might have $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$.

Then $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$.

Example. Consider $u = x^4 y + y^2 z^3$ where $x = rse^t$, $y = rs^2 e^{-t}$, $z = r^2 s(\sin t)$. Find $\frac{\partial u}{\partial s}$.

In full generality: If u is a function of $x_1, x_2, \ldots, x_n$ and each x_j is a function of $t_1, t_2, \ldots, t_m$, then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}.$$

Implicit differentiation

- Simplify implicit differentiation calculations!

Involving two variables

Consider $F(x, y) = C$ like

Implicitly y is a function of x:

$$F\left(x, f(x)\right) = C$$

Differentiating w.r.t. x:

$$F(x, y) \quad = C$$

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Involving three variables

Consider $F(x, y, z) = C$

Implicitly z is a function of x and y:

$$F\left(x, y, f(x, y)\right) = C$$

Differentiating w.r.t. x:

$$F(x, y, z) \qquad = C$$

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} = 0$$

Solving for $\frac{\partial z}{\partial x}$ gives

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

Example. Find $\frac{\partial z}{\partial x}$ if $x^3 + y^2 + z^3 + 6xyz = 1$

Definition of the directional derivative

Partial derivatives allow us to see how fast a function changes.

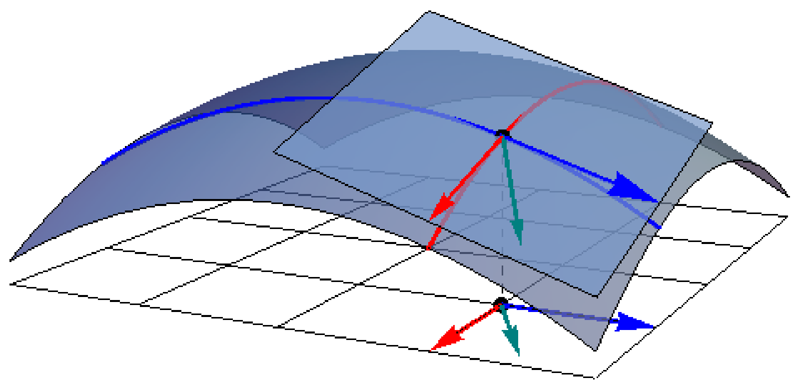
$D_x f = f_x(x, y)$ is the rate of change of f in the x-direction. Toward $\vec{\mathbf{i}} = (1, 0)$

$D_y f = f_y(x, y)$ is the rate of change of f in the y-direction. Toward $\vec{\mathbf{j}} = (0, 1)$

Question: How fast is $f(x, y)$ changing in **some other direction**?

What does that even mean?

Question: What is the rate of change of f toward *unit vector* $\vec{\mathbf{u}} = (a, b) = (\cos\theta, \sin\theta)$?



Definition: The directional derivative of f in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b.$$

Directional derivative example

Example. Find $D_{\vec{\mathbf{u}}}f$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec{\mathbf{u}}$ is the unit vector in the xy-plane at angle $\theta = \pi/6$.

Answer: First, find the vector $\vec{\mathbf{u}} =$

Next, find the partial derivatives:

$$\frac{\partial f}{\partial x} = \qquad\qquad \frac{\partial f}{\partial y} = \qquad\qquad .$$

We conclude that $D_{\vec{\mathbf{u}}}f(x,y) =$

Example. Calculate $D_{\vec{\mathbf{u}}}f(1,2)$ and interpret this answer.

$$\begin{aligned} D_{\vec{\mathbf{u}}}f(1,2) &= (3\cdot 1 - 3\cdot 2)\tfrac{\sqrt{3}}{2} + (-3\cdot 1 + 8\cdot 2)\tfrac{1}{2} \\ &= \frac{13 - 2\sqrt{3}}{2} \approx 3.9 \end{aligned}$$

Interpretation: One unit step in the $\vec{\mathbf{u}}$ direction increases $f(x,y)$ by approximately 3.9 units.