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Example. Let $z = x^2y + 3xy^2$, where $x = \sin 2t$, $y = \cos t$. Find $\frac{dz}{dt}$, $z'(0)$.
Answer:

More Chains

Alternatively, we might have $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$.

Then $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$.

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Example. Consider $u = x^4 y + y^2 z^3$ where $x = rse^t$, $y = rs^2 e^{-t}$, $z = r^2 s(\sin t)$. Find $\frac{\partial u}{\partial s}$.

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In full generality: If u is a function of $x_1, x_2, \ldots, x_n$ and each x_j is a function of $t_1, t_2, \ldots, t_m$, then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}.$$

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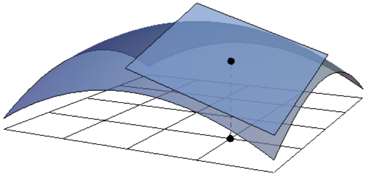
Example. Find $\frac{\partial z}{\partial x}$ if $x^3 + y^2 + z^3 + 6xyz = 1$

Definition of the directional derivative

Partial derivatives allow us to see how fast a function changes.

$D_x f = f_x(x, y)$ is the rate of change of f in the x-direction.

$D_y f = f_y(x, y)$ is the rate of change of f in the y-direction.



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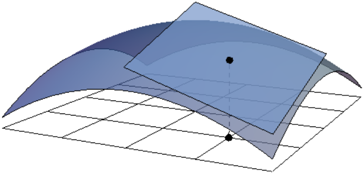

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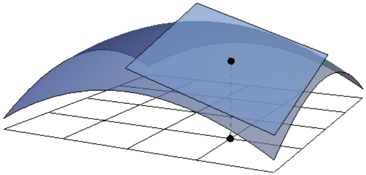

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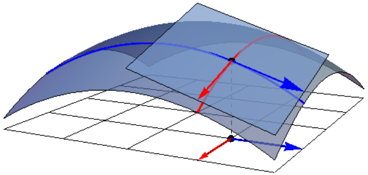

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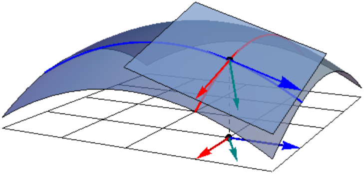

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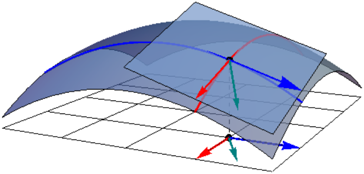

Definition: The directional derivative of f in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b.$$

Directional derivative example

Example. Find $D_{\vec{\mathbf{u}}} f$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\vec{\mathbf{u}}$ is the unit vector in the xy-plane at angle $\theta = \pi/6$.

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Interpretation: One unit step in the $\vec{\mathbf{u}}$ direction increases $f(x,y)$ by approximately 3.9 units.