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The gradient is also defined for functions of more than two variables.
For example, for a function of three variables, $f(x, y, z)$,

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \vec{\mathbf{i}} + f_y \vec{\mathbf{j}} + f_z \vec{\mathbf{k}}$$

$$\text{and } D_{\vec{\mathbf{u}}} f = \nabla f \cdot \vec{\mathbf{u}}$$

Applying ∇f

Example. Let $f(x, y, z) = x \sin(yz)$. Find the directional derivative of f at $(1, 3, 0)$ in the direction $\vec{\mathbf{v}} = \vec{\mathbf{i}} + 2\vec{\mathbf{j}} - \vec{\mathbf{k}}$.

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Interpretation?

An important interpretation of the gradient

Question: Given a function $f(x, y)$ and a point (x_0, y_0),

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Consequence: ∇f represents the direction of fastest increase of f.

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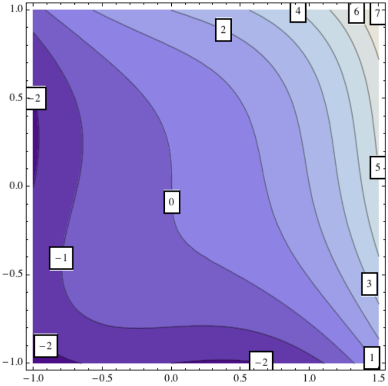

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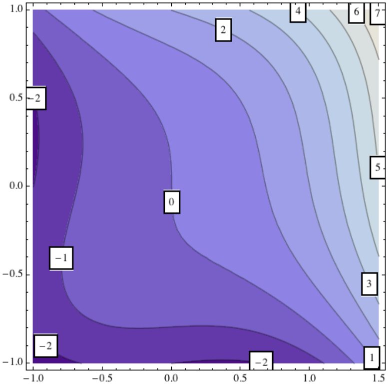

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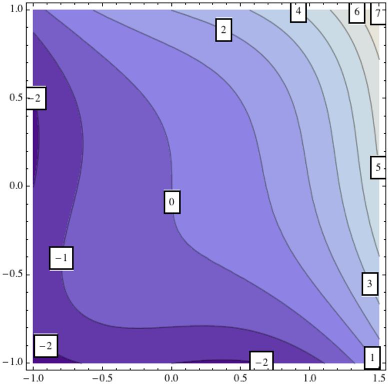
♡ Connecting along this path gives ♡
♡ the **path of steepest ascent**. ♡
Chloe says "hi".



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which means $\nabla F \perp \vec{\mathbf{r}}'(t) = 0$.