Local and Global Extrema (Relative and Absolute)

$f(x)$ has a **local maximum** at $x = a$
if for all points x near a, $f(x) \leq a$.

$f(x)$ has a **local minimum** at $x = a$
if for all points x near a, $f(x) \geq a$.

If $f(x)$ has a local max or local min at $x = a$,
then $f'(a) = 0$ or $f'(a)$ does not exist.
This is called a **critical point**. (pics!)

However, **If** $f'(a) = 0$ or $f'(a)$ DNE
then this **does not imply** that $x = a$
has a local max or a local min. (pics!)

Extreme Value Theorem: If f is continuous on
a closed interval (meaning: ____________________),
then $f(x)$ attains its absolute max and absolute min
somewhere on this interval.

Determining relative extrema

Important vocabulary:

- A maximum or minimum: means ________________.
- A maximum value or minimum value: means ________.

We can try to determine if a critical point is a local extremum using:

The second derivative test.

If the second partial derivatives of $f(x, y)$ are continuous around (a, b) **And if** $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then define $D(a, b)$:

$$D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

1. If $D > 0$ and $f_{xx} > 0$, then (a, b) is a local minimum.
2. If $D > 0$ and $f_{xx} < 0$, then (a, b) is a local maximum.
3. If $D < 0$, then (a, b) is a **saddle point** of f.
4. If $D = 0$, the test is inconclusive.

Extreme Examples

Example. Find the local extrema and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

- Critical points:
- For each: Find $D(a, b)$, classify.

Example. Find the global extrema of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $0 \le x \le 3$ and $0 \le y \le 2$.

- Critical points on interior
- Check boundary

Optimization is just finding maxima and minima

Example. A rectangular box with no lid is made from $12\,\mathrm{m}^2$ of cardboard. What is the maximum volume of the box?

Solution. Let length, width, and height be x, y, and z, respectively. Then the question asks us to maximize $V =$ __________, subject to ________________.

Solving for z gives $z = \frac{12-xy}{2x+2y}$. Inserting, $V = xy\left(\frac{12-xy}{2x+2y}\right)$.

To find an optimum value, solve for $\frac{\partial V}{\partial x} = 0$ and $\frac{\partial V}{\partial x} = 0$.

$\frac{\partial V}{\partial x} = 0 \rightsquigarrow$

$\frac{\partial V}{\partial y} = 0 \rightsquigarrow$

Solving these simultaneous equations, $12 - 2xy = x^2 = y^2 \Rightarrow x = \pm y$. Because this is real world, ______, so we solve $12 - 3x^2 = 0$: ______.

This problem must have an absolute maximum, which must occur at a critical point. (Why?) Therefore $(x, y, z) = (2, 2, 1)$ is the absolute maximum, and the maximum volume is $xyz = 4$.