Optimization subject to constraints

The method of Lagrange multipliers is an alternative way to find maxima and minima of a function $f(x, y, z)$ subject to a given constraint $g(x, y, z) = k$.

Motivating Example. Suppose you are trying to find the maximum and minimum value of $f(x, y) = y - x$ when we only consider points on the curve $g(x, y) = x^2 + 4y^2 = 36$.

What should we do?

The method of Lagrange multipliers

To find the maxima and minima of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (as long as $\nabla g \neq 0$ on this constraint)

- Solve for all tuples (x, y, z, λ) such that

$$\nabla f(x, y, z) = \lambda \cdot \nabla g(x, y, z) \quad \textbf{and} \quad g(x, y, z) = k$$

  - (Solve this system of four equations and four unknowns.)
  - *In words:* the gradient of f is parallel to the gradient of g.

- Evaluate f at all points (x, y, z) you find.
  - The largest f value corresponds to a maximum
  - The smallest f value corresponds to a minimum.

- λ is called a Lagrange multiplier.
- Careful about when this applies. ($\nabla g \neq 0$)

Optimization Example, revisited

Example. A rectangular box with no lid is made from 12 m^2 of cardboard. What is the maximum volume of the box?

Goal: Maximize $V = xyz$ subject to $g(x, y, z) = 2xz + 2yz + xy = 12$.

By the method of Lagrange multipliers, we need to solve:

$\langle yz, xz, xy\rangle = \lambda\langle 2z+y, 2z+x, 2x+2y\rangle$ **and** $2xz+2yz+xy = 12$

Solve: $$\begin{cases} yz = \lambda(2z + y) \\ xz = \lambda(2z + x) \\ xy = \lambda(2x + 2y) \\ 2xz + 2yz + xy = 12 \end{cases}$$

- Four equations, four unknowns, so possibly solvable.
- Eliminate λ using first two equations. (& that $\lambda \neq 0$ by Eq. (4).)
- Multiply Eq. (2) by y, Eq. (3) by z, simplify.

Why does this work?

For functions of two variables:

The tangent line to the curve $g(x, y) = k$ and the level curve $f(x, y) = \max$ are parallel, so their normals are too. We conclude that $\nabla f(x, y) = \lambda \nabla g(x, y)$.

For functions of three variables:

The tangent plane to the surface $g(x, y, z) = k$ and the level surface $f(x, y, z) = \max$ are parallel, so their normals are too. We conclude that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

Another example

Example. Find the extreme values of $f(x,y) = x^2 + 2y^2$ in the region $x^2 + y^2 \leq 1$.

Game plan:

- Check for critical points on the interior of the region.

For critical points, solve $f_x = 0$, $f_y = 0$:

What is $f(x,y)$ there?

- Use Lagrange multipliers to find maxs, mins on boundary.

Find x, y, λ satisfying $\nabla f = \lambda \nabla g$ and $x^2 + y^2 = 1$:

What is $f(x,y)$ there?

Solution?