Optimization subject to constraints

The method of Lagrange multipliers is an alternative way to find maxima and minima of a function $f(x, y, z)$ subject to a given constraint $g(x, y, z) = k$.

Motivating Example. Suppose you are trying to find the maximum and minimum value of $f(x, y) = y - x$ when we only consider points on the curve $g(x, y) = x^2 + 4y^2 = 36$.

What should we do?

The method of Lagrange multipliers

To find the maxima and minima of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (as long as $\nabla g \neq 0$ on this constraint)

- Solve for all tuples (x, y, z, λ) such that

$$\nabla f(x, y, z) = \lambda \cdot \nabla g(x, y, z) \quad \textbf{and} \quad g(x, y, z) = k$$

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- Careful about when this applies. $(\nabla g \neq 0)$

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- Eliminate λ using first two equations. (& that $\lambda \neq 0$ by Eq. (4).)
- Multiply Eq. (2) by y, Eq. (3) by z, simplify.

Why does this work?

For functions of two variables:

The tangent line to the curve $g(x, y) = k$ and the level curve $f(x, y) = \max$ are parallel, so their normals are too. We conclude that $\nabla f(x, y) = \lambda \nabla g(x, y)$.

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For functions of three variables:

The tangent plane to the surface $g(x, y, z) = k$ and the level surface $f(x, y, z) = \max$ are parallel, so their normals are too. We conclude that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

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Solution?