Riemann Sum Comparison

Riemann sum to approximate area

Subdivide $[a, b]$ into n intervals I.

Interval width: $\Delta x_i = x_i - x_{i-1}$.

Choose sample point x_i^* in each I

The Riemann sum
$$\sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

approximates area under curve.

Integral is limit of Riemann sums.

$$\int_a^b f(x) \, dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i \quad \text{(If exists)}$$

Riemann Sum to approximate volume
(Today: Rectangles $[a, b] \times [c, d]$.)

Subdivide $[a, b]$ into m intervals.
Subdivide $[c, d]$ into n intervals.

Each subrect. $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ has area $\Delta A_{ij} = \Delta x_i \Delta y_j$.

The Riemann sum
$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_i^*) \Delta A_{ij}$$

approximates volume under surface.

The **double integral** is limit of **R.S.**

$$\iint_R f(x, y) \, dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \Delta y_i \to 0}} \textbf{R.S.} \quad \text{(If exists)}$$

Notes and an example

▶ If f is continuous, the limit always exists!

▶ The volume under the surface is approximated by adding up the volumes of a bunch of rectangular prisms.

Example. Estimate the volume that lies above the square $R = [0, 2] \times [0, 2]$ and below the surface $z = 16 - x^2 - 2y^2$ by calculating a Riemann sum with four terms.

Answer: The region R can be divided into four squares, each with area $\Delta A =$ _____.

Choose a sample point in each region.
(Ex: Upper right corner.)

Volume $\approx f(1, 1)\Delta A_{11} + f(1, 2)\Delta A_{12} + f(2, 1)\Delta A_{21} + f(2, 2)\Delta A_{22}$
$$=$$

Calculating double integrals

We never calculate a double integral by taking limits of Riemann sums.

Instead, we use the method of iterated integrals:

Calculate the volume by adding slices together.
For a fixed x, we can find the area of an
(infinitesimal) slice of $f(x, y)$ by integrating
over y. Then we can integrate these areas
over all slices to give the volume.

Area of slice: $A(x) = \displaystyle\int_{y=c}^{y=d} f(x, y)\, dy$ $[\textit{Think of } x \textit{ as constant.}]$

Volume: $V = \displaystyle\int_{x=a}^{x=b} A(x)\, dx = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x, y)\, dy \right] dx.$

We write $\displaystyle\int_{a}^{b} \int_{c}^{d} f(x, y)\, dy\, dx.$

The order of the dy and dx tells you which to integrate first.
Work from the inside out.

Fubini's Theorem

If f is continuous on the rectangle $R = [a, b] \times [c, d]$ then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Also true in general if

- ▶ f is bounded on R

- ▶ f discontinuous only on a finite number of smooth curves

- ▶ the iterated integrals exist

Intuition: In our picture, we could have sliced the volume by fixing y instead of x.

Take away message: When f is nice, we can choose the order of integration to make our life easier.

Double integrals

Example. Find $\iint_R y \sin(xy)\, dA$ where $R = [1, 2] \times [0, \pi]$.

Is it easier to integrate with respect to _____.

Properties of double integrals

When $f(x, y)$ is a product of (a fcn of x) and (a fcn of y) over a rectangle $[a, b] \times [c, d]$, then the double integral decomposes nicely:

$$\iint_R g(x)h(y)\, dA = \left[\int_a^b g(x)\, dx \right] \cdot \left[\int_c^d h(y)\, dy \right]$$

For general regions R: (not necessarily rectangles)

▶ $\iint_R (f + g)\, dA = \iint_R f\, dA + \iint_R g\, dA$

▶ $\iint_R cf\, dA = c \iint_R f\, dA$

▶ If $f(x, y) \geq g(x, y)$ for all $(x, y) \in R$, then $\iint_R f\, dA \geq \iint g\, dA$.