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The **double integral** is limit of **R.S.**

$$\iint_R f(x, y)\,dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \Delta y_i \to 0}} \textbf{R.S.}$$ (If exists)

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$$\begin{aligned}\text{Volume} &\approx f(1,1)\Delta A_{11} + f(1,2)\Delta A_{12} + f(2,1)\Delta A_{21} + f(2,2)\Delta A_{22} \\ &= \end{aligned}$$

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The order of the dy and dx tells you which to integrate first.
Work from the inside out.

Fubini's Theorem

If f is continuous on the rectangle $R = [a, b] \times [c, d]$ then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

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Take away message: When f is nice, we can choose the order of integration to make our life easier.

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Properties of double integrals

When $f(x, y)$ is a product of (a fcn of x) and (a fcn of y) over a rectangle $[a, b] \times [c, d]$, then the double integral decomposes nicely:

$$\iint_R g(x)h(y)\,dA = \left[\int_a^b g(x)\,dx\right] \cdot \left[\int_c^d h(y)\,dy\right]$$

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- $\iint_R cf\, dA = c \iint_R f\, dA$
- If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then $\iint_R f\, dA \geq \iint g\, dA$.