General regions *ARE* rectangles

Last time: $\iint_R f(x, y)\, dy\, dx$ when R is a rectangle. (Riemann)

Question: Does $\iint_D f(x, y)\, dy\, dx$ make sense when domain D is not a rectangle?

Answer: Yes, because we can view $\iint_D$ as a $\iint_R$:

Suppose D is not a rectangle. Then *fit* D in a rectangle R, and extend the function $f(x, y)$ to be defined over all R:

$$F(x, y) = \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \notin D \end{cases}$$

Then define $$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA.$$

(Which we know exists)

Calculating double integrals over non-rectangles

The way we decide to integrate $\iint_D$ depends on the shape of D:

Type I

If D is defined by $\left\{ \begin{array}{r} \text{an “upper function” } y = g_2(x) \\ \text{a “lower function” } y = g_1(x) \end{array} \right\}$,

then integrate by slices with fixed x values.

$$\iint_D f(x,y)\,dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

Type II

If D is defined by $\left\{ \begin{array}{r} \text{a “left function” } x = h_2(y) \\ \text{a “right function” } x = h_1(y) \end{array} \right\}$,

then integrate by slices with fixed y values.

$$\iint_D f(x,y)\,dA = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dy\,dx$$

Determine type by looking at which slices cut all the way through D.
Some regions work either way. Choose based on $f(x,y)$.

Simple Example

Example. Find $\iint_D (x + 2y)\, dA$,
where D is bounded by $y = 2x^2$ and $y = 1 + x^2$.

Steps:

1. Plot the curves (Draw a picture!)
2. Find points of intersection
3. Determine order of integration
4. Determine "upper" and "lower" functions, other bounds
5. Do the integrals.

Not-as-simple Example

Example. Find $\iint_D xy\, dA$, where D bdd by $y = x - 1$ and $y^2 = 2x + 6$.
Important: Draw a picture.

If we were to set up the integral as slices in x, there would be two different lower functions, depending on whether $x \leq 1$ or $x \geq 1$. This would require doing **two** integrals! (What are they?)

Instead, integrate with slices in y. The "upper" function is ________ and the "lower" function is __________.

We calculate $\displaystyle\int_{y=-2}^{y=4} \int_{x=\frac{y^2-6}{2}}^{x=y+1} xy\, dx\, dy = \int_{y=-2}^{y=4} \left[y\frac{x^2}{2} \right]_{x=\frac{y^2-6}{2}}^{x=y+1} dy =$

$$\frac{1}{2}\int_{y=-2}^{y=4} y(y+1)^2 - y\left(\frac{y^2-6}{2}\right)^2 dy = \frac{1}{2}\int_{y=-2}^{y=4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \tfrac{1}{2}\left[-\tfrac{y^6}{24} + y^4 + \tfrac{2}{3}y^3 - 4y^2 \right]_{y=-2}^{y=4} = 36$$

A Wordy Example

Sometimes you need to find D and f from the problem statement.

Example. Set up the integral that finds the volume of the solid bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution. Use the planes to understand and draw the solid.
Project the solid onto xy-plane to find domain D.
Where does $x + 2y + z = 2$ intersect the axes?
(Draw in 3-space and on xy-plane.)

What does $z = 0$ do? What does $x = 0$ do?

What does $x = 2y$ do?

So our domain D looks like:
(intersection pts? slicing direction? start/stop?)

Our function is $f(x, y) = z = 2 - x - 2y$, and our integral is

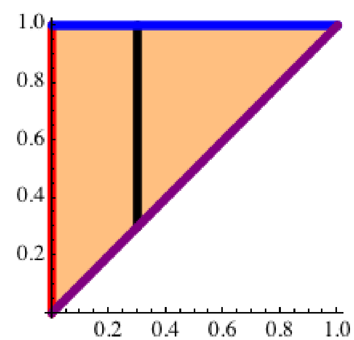
$$\int_{x=0}^{x=1} \int_{y=x/2}^{y=1-x/2} (2 - x - 2y)\, dy\, dx$$

Changing the order of integration

We might want to change the order of integration in iterated integrals.

Caution: For non-rectangles, we have to be very careful!

$$\int_0^1 \int_{y=x}^{y=1} f(x,y)\,dy\,dx \quad \rightsquigarrow$$



$$\rightsquigarrow \quad \int_0^1 \int_{x=0}^{x=y} f(x,y)\,dx\,dy$$

When chopping in x,

$$\left\{ \begin{array}{c} x \text{ varies from 0 to 1,} \\ \text{upper fcn is } y = 1 \\ \text{lower fcn is } y = x \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} y \text{ varies from 0 to 1,} \\ \text{upper fcn is } x = y \\ \text{lower fcn is } x = 0 \end{array} \right\}$$

When chopping in y,

Double integral properties

Property. Suppose that $D = D_1 \cup D_2$ (where D_1 and D_2 don't overlap). Then

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA.$$

Consequence: Break down complicated regions into Type I and Type II regions.

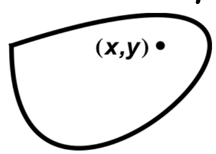
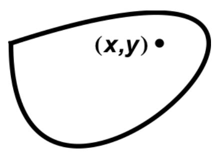
Property. Suppose $m \le f(x, y) \le M$ for all $(x, y) \in D$. Then

$$m \cdot \text{Area}(D) \le \iint_D f(x, y)\,dA \le M \cdot \text{Area}(D)$$

Consequence: This gives a crude approximation for the integral.

Application: Density

Suppose you have a 2D sheet of metal (a **lamina**) where density varies over the sheet.

mass density function $\rho(x,y)$ (mass per unit area) 	The total mass of the object is $$m = \iint_D \rho(x,y)\,dA$$
charge density function $\sigma(x,y)$ (charge per unit area) 	The total charge on the object is $$Q = \iint_D \sigma(x,y)\,dA$$

Example. Find the mass of a △ lamina w/ corners $(1,0)$, $(0,2)$, $(1,2)$, and mass density function $\rho(x,y) = 1 + 3x + y$.

Solution. $m = \int_{x=0}^{x=1} \int_{y=2-2x}^{y=2} (1 + 3x + y)\,dy\,dx$

$= \int_{x=0}^{x=1} (y + 3xy + \frac{y^2}{2})\Big|_{y=2-2x}^{y=2}\,dx$

$= \int_{x=0}^{x=1} (6x + 4x^2)\,dx = 3x^2 + \frac{4}{3}x^3\Big]_{x=0}^{x=1} = \frac{13}{3}$