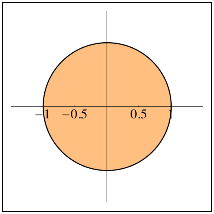
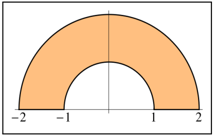
Some regions *demand* polar coordinates

Some regions are best described in polar coordinates:



Parts of circles



Parts of annuli

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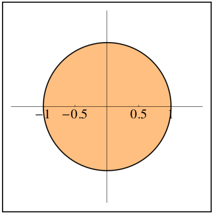

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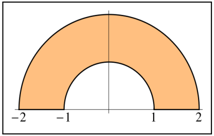

Parts of annuli

These are called **polar rectangles** because they have the form

$$\left\{ \begin{array}{c} a \leq r \leq b \\ \alpha \leq \theta \leq \beta \end{array} \right\} \text{ for constants } \left\{ \begin{array}{c} a,\ b \\ \alpha,\ \beta \end{array} \right\}$$

Polar coordinates

Goal: Convert a double integral involving x's and y's into a double integral involving r's and θ's.

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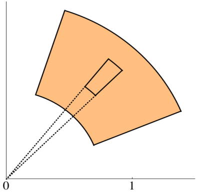

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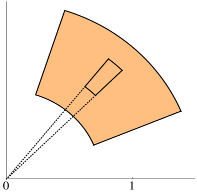

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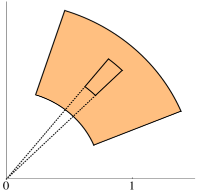

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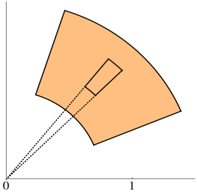

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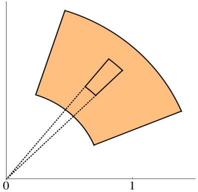

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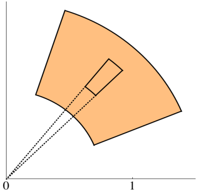

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Consequence: To calculate $\iint_D f(x, y)\,dA$,
when D is best described in polar coordinates, calculate

$$\iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

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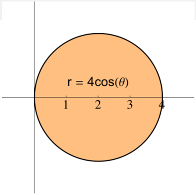

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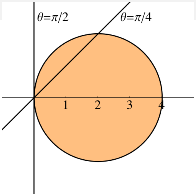

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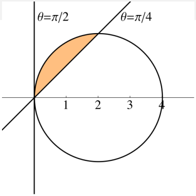

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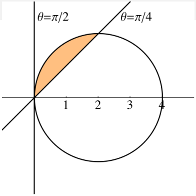

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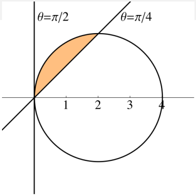

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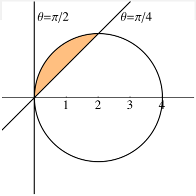

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