Triple Integrals

For a function $f(x, y, z)$ defined over a bounded region E in three dimensions, we can take the triple integral

$$\iiint_E f(x, y, z)\, dV$$

If f is continuous over a region that is a **box**

$$B = [a, b] \times [c, d] \times [r, s],$$

Fubini's theorem says that

$$\iiint_B f(x, y, z)\, dV = \int_{z=r}^{z=s} \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y, z)\, dx\, dy\, dz$$

And that you are allowed to choose the order of integration you wish.

Example

Example. Find $\iiint_B xyz^2\, dV$ for $B = [0, 1] \times [-1, 2] \times [0, 3]$

Solution.

The difficulties arise when

- Regions are not boxes. (Today)
- Regions are best defined in polar-like coordinates. (Next time)

Setting up complicated triple integrals

- We only consider triple integrals over regions that can be defined as being between two surfaces.
- This allows us to reduce our triple integral to a double integral. (Which may itself be complicated...)

Three types:

Type 1
$$\iiint_E f\, dV = \iint_D \left[\int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\, dz\right] dA$$

Type 2
$$\iiint_E f\, dV = \iint_D \left[\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\, dx\right] dA$$

Type 3
$$\iiint_E f\, dV = \iint_D \left[\int_{y=u_1(x,z)}^{y=u_2(x,z)} f(x,y,z)\, dy\right] dA$$

Triple Integral Strategies

The hard part is figuring out the bounds of your integrals.

- Project your region E onto one of the xy-, yz-, or xz-planes, and use the boundary of this projection to find bounds on domain D.
- Over this domain D, the region E is defined by some "higher function" and some "lower function". These give the bounds on the innermost integral.

You may need to try multiple projections to find the easiest integral to integrate. Then use all the tools in your toolbox to integrate it.

Example

Example. Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$ where E is the region bounded by the hyperboloid $y = x^2 + z^2$ and the plane $y = 4$.

Project onto xy-plane

D is defined by

The higher and lower functions are

$$\iiint_E \sqrt{x^2+z^2}\,dV =$$

$$\int_{-2}^{2}\int_{x^2}^{4}\left(\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\right) dy\,dx$$

Project onto xz-plane

D is defined by

The higher and lower functions are

$$\iiint_E \sqrt{x^2+z^2}\,dV =$$

$$\iint_{\substack{D=\text{circle}\\ x^2+z^2=4}}\left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right) dA$$

Example, continued

Now calculate $\displaystyle\iint_{\substack{D=\text{circle}\\x^2+z^2=4}} \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right) dA$

There is no y, so the innermost integral is easy:

$$= \iint_{\substack{D=\text{circle}\\x^2+z^2=4}} \left(\sqrt{x^2+z^2} \cdot y \right]_{x^2+z^2}^{4} dA$$

$$= \iint_{\substack{D=\text{circle}\\x^2+z^2=4}} \sqrt{x^2+z^2} \cdot \left(4 - x^2 - z^2\right) dA$$

This integral is easier to do using ________________

Loose ends

Density in three dimensions

- Given a mass density function $\rho(x, y, z)$ (mass per unit volume)

$$\text{mass} = \iiint_E \rho(x, y, z)\, dV.$$

Average value in three dimensions

- The average value of a function $f(x, y, z)$ over a region E is

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$