

Generalizations of Polar Coordinates

When 2-dim'l regions D have radial flavors, we use polar coordinates.

When 3-dim'l regions E have radial flavors, there are two choices:

Cylindrical coordinates

A point can have coords (r, θ, z) :

(r, θ) are polar coords of xy -plane
(r is the distance from the z -axis)
and z is the “distance” to xy -plane

Useful when problems involve
symmetry about an axis. Cylinder,
Paraboloids, Cones w/flat bases

Spherical coordinates

A point can have coords (ρ, θ, ϕ) .

ρ is the distance from the origin
 θ is same as in polar
 ϕ is \angle between $+z$ axis and \overline{OP} .

Useful when problems involve
symmetry about a point
Spheres, Cones with curved bases.

Converting from cartesian coordinates

Cylindrical coordinates

Conversion:

$$\begin{aligned}x &= r \cos \theta & y &= r \sin \theta & z &= z \\r^2 &= x^2 + y^2 & \tan \theta &= \frac{y}{x} & z &= z\end{aligned}$$

$$dV = r \, dr \, d\theta \, dz$$

Spherical coordinates

Conversion:

$$\begin{aligned}x &= \rho \sin \phi \cos \theta & z &= \rho \cos \phi \\y &= \rho \sin \phi \sin \theta & \rho^2 &= x^2 + y^2 + z^2; \\ \tan \phi &= \frac{\sqrt{x^2 + y^2}}{z}\end{aligned}$$

$$dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

Practice

Cylindrical coordinates

Practice changing coordinates:

$$(r, \theta, z) = (2, \frac{2\pi}{3}, 1); (x, y, z) = (3, -3, 7)$$

Identify cyl. coord. equations:

2. $r = 2; z = r^2; r^2 - 2z^2 = 4$

3. Sketch $r^2 \leq z \leq 2 - r^2$

Convert to cylindrical coordinates

4. $x^2 + y^2 + z^2 = 2; x^2 + y^2 = 2y$

5. Give solid between $x^2 + y^2 = 1$
and $x^2 + y^2 + z^2 = 4$.

6.
$$\left\{ \begin{array}{l} -2 \leq x \leq 2 \\ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} \\ \sqrt{x^2+y^2} \leq z \leq 2 \end{array} \right\}$$

Spherical coordinates

Practice changing coordinates:

$$(\rho, \theta, \phi) = (2, \frac{\pi}{4}, \frac{\pi}{3}); (x, y, z) = (-1, 1, \sqrt{6})$$

Identify sph. coord. equations:

2. $\phi = \frac{\pi}{3}; \rho \sin \phi = 2; \rho = 2 \cos \phi$

3. Sketch $(2 \leq \rho \leq 3 \ \& \ \frac{\pi}{2} \leq \phi \leq \pi)$
Sketch $(0 \leq \phi \leq \frac{\pi}{3} \ \& \ \rho \leq 2)$

Convert to spherical coordinates

4. $z = x^2 + y^2; z = x^2 - y^2$

5. Give solid inside $x^2 + y^2 + z^2 = 4$,
above xy -plane, below $z = \sqrt{x^2 + y^2}$.

6.
$$\left\{ \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq \sqrt{1-x^2} \\ \sqrt{x^2+y^2} \leq z \leq \sqrt{2-x^2-y^2} \end{array} \right\}$$