Course Notes

Multivariable Calculus, Spring 2014

Queens College, Math 201

Prof. Christopher Hanusa

http://qcpages.qc.edu/~chanusa/courses/201sp14/

Class Introductions

Arrange yourselves into groups of four or five people,
With people you **don't know**.

- Introduce yourself. (your name, where you're from, your major)
- What brought you to this class?
- Fill out **the front of** your notecard:
  - Write your name. (Stylize if you wish.)
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- **Discuss!** What is "Calculus"?
  Brainstorm then organize into a mind map.
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- **Discuss!** What is "Calculus"?
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- How do these ideas translate to multivariable calculus?

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All homeworks posted online; first one (many parts) due Wednesday.

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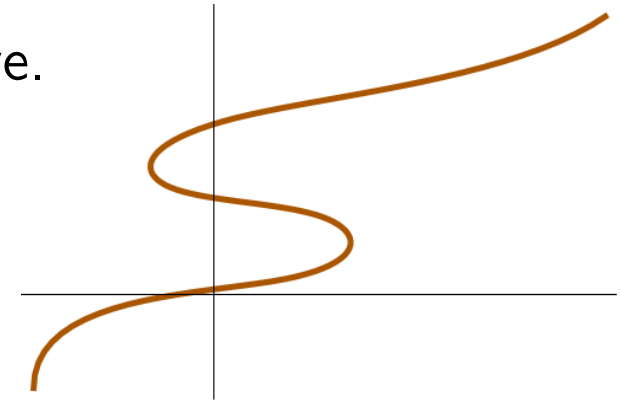
Parametric Curves

Imagine a particle traveling along this curve.



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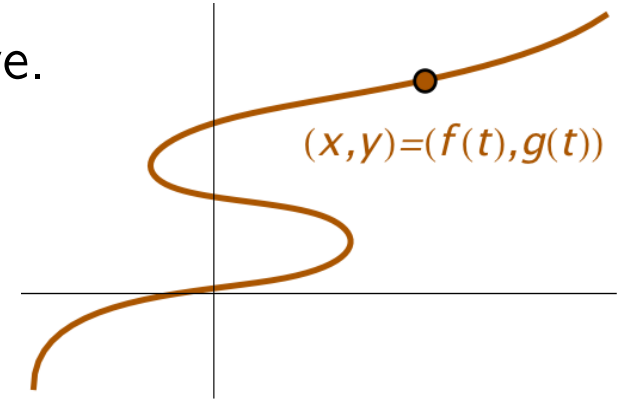

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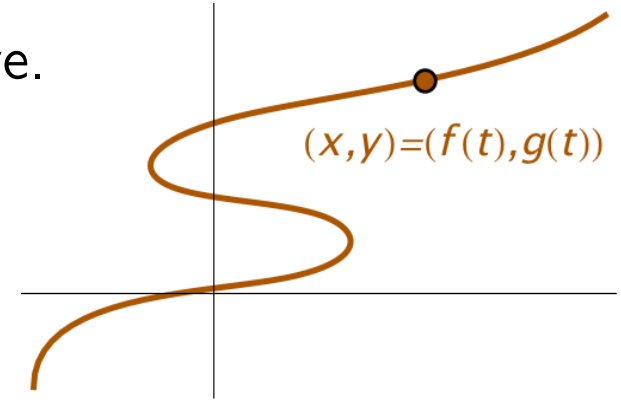

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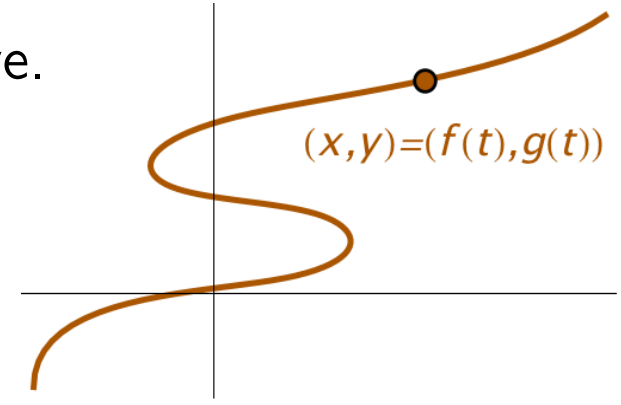

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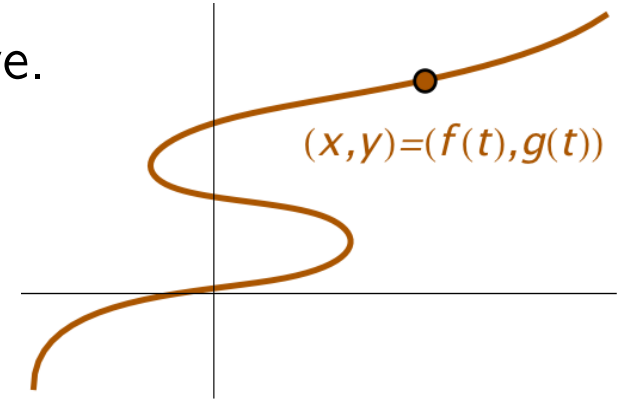

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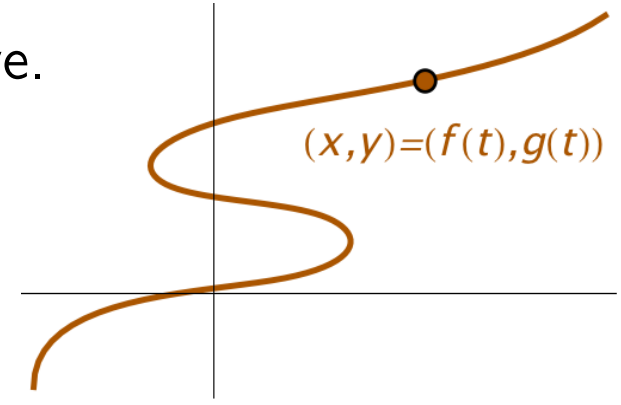

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Goal 1: Understand parametric curves. (Today)

Goal 2: Do calculus using parametric curves. (Next time)

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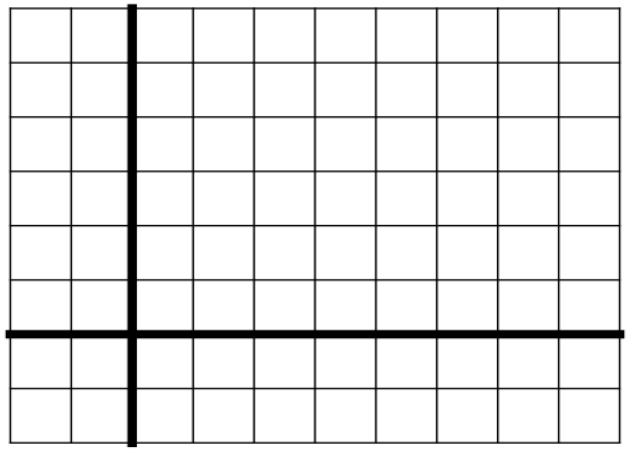

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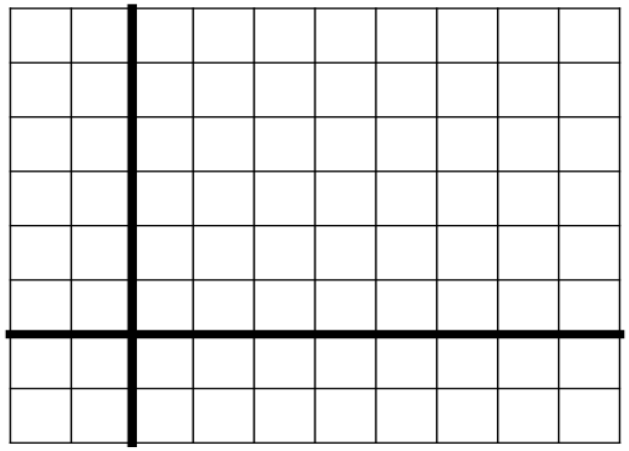

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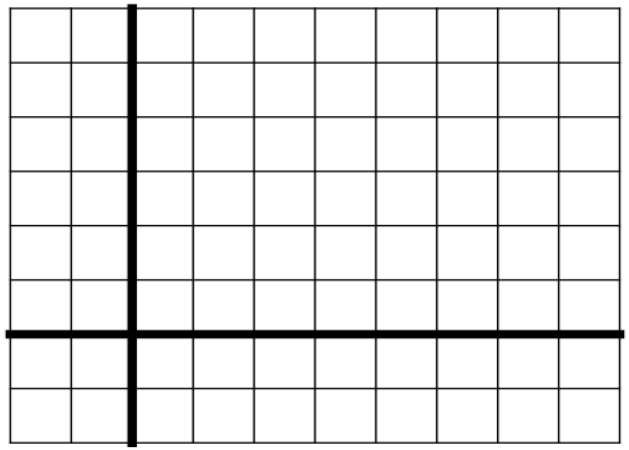

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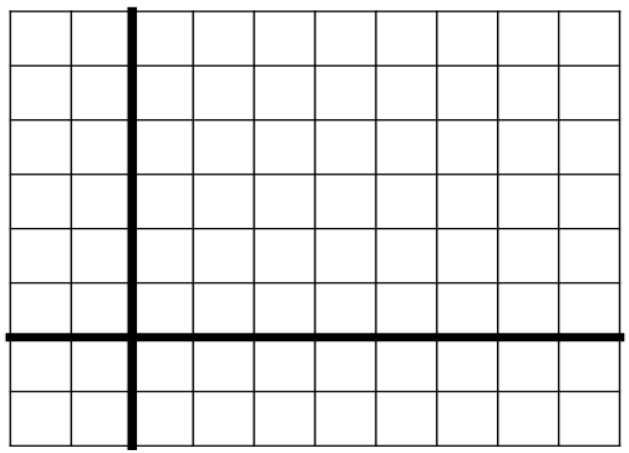

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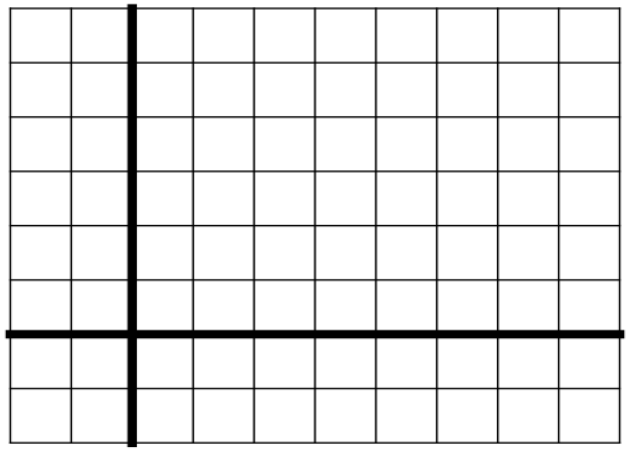

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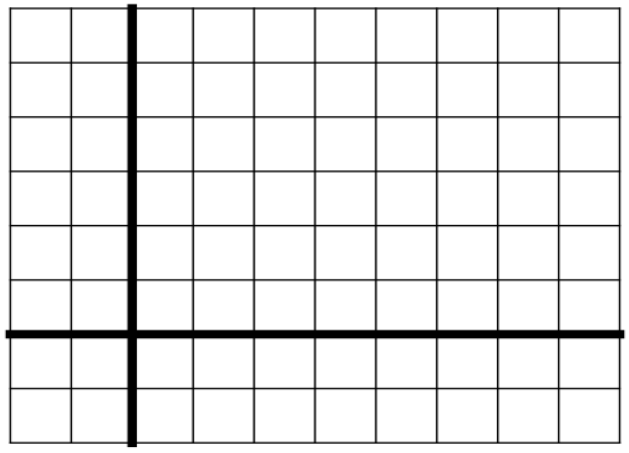

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Solve for t in second equation: $t = y - 1$ and plug in:
$x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$, a "sideways parabola".

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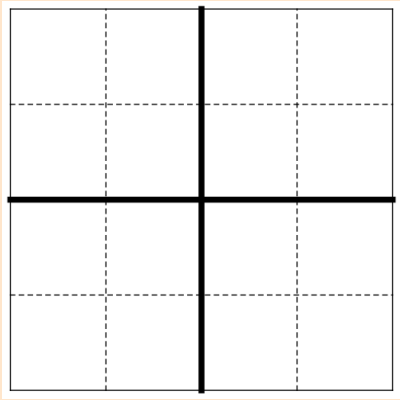

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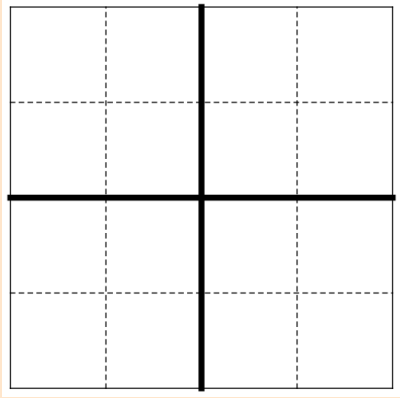

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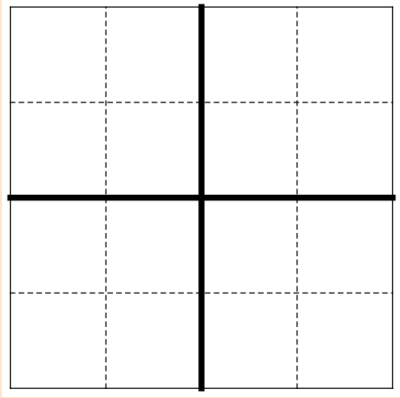

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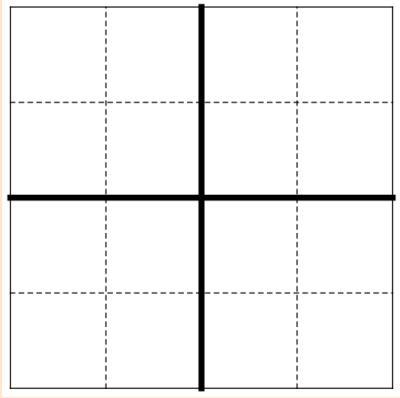

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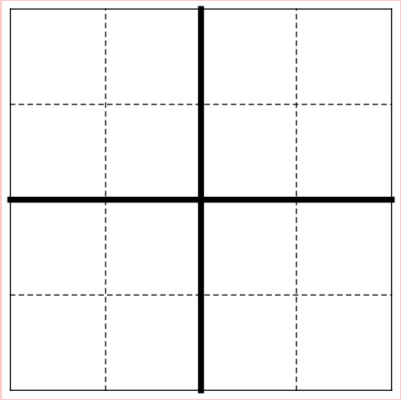

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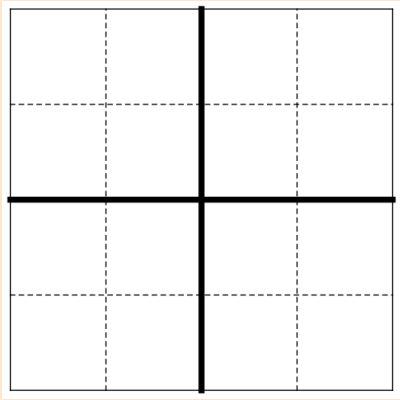

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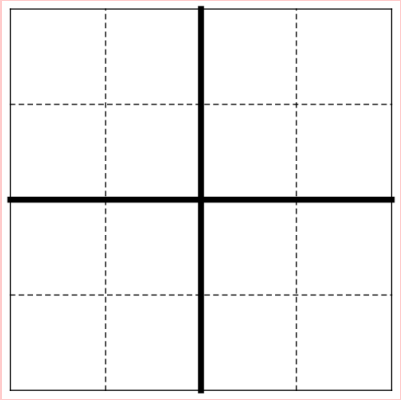

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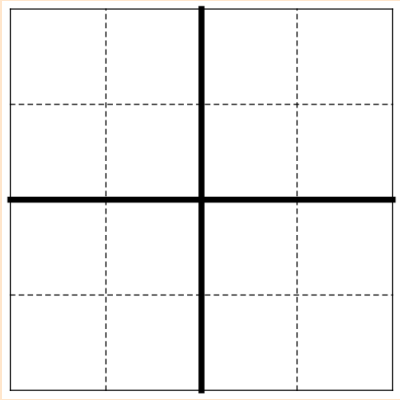

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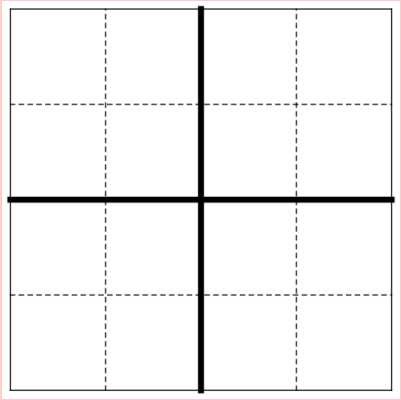

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You need to know your trig functions and values at certain points!!!!

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In other words, use the parametric equations

$$x(t)=r\cos t+h \quad \text{and} \quad y(t)=r\sin t+k.$$

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If we want to draw a circle at some other place

$$(x-h)^2+(y-k)^2=r^2,$$

set $x-h=r\cos t$ and $y-k=r\sin t$.

In other words, use the parametric equations

$$x(t)=r\cos t+h \quad \text{and} \quad y(t)=r\sin t+k.$$

Try it out! Get out your graphing calculator TI-(≤ 86).

Switch to Parametric mode: `MODE` ↓ ↓ ↓ `PAR` (`Enter`).

Enter the equations X_1 `= 3 cos(T) + 2` and Y_1 `= 3 sin(T) + 4`.

Set the domain of T to be from 0 to 2π.

This plots a circle of radius 3 centered at $(2,4)$.

Computers to the rescue

Calculators and computers can graph much more complicated curves.

$$x_1(t) = t + 2\sin(2t) \quad \text{and} \quad y_1(t) = t + 2\cos(5t)$$

$$x_2(t) = 1.5\cos t - \cos 30t \quad \text{and} \quad y_2(t) = 1.5\sin t - \sin 30t$$

$$x_3(t) = \sin(t + \cos 100t) \quad \text{and} \quad y_3(t) = \cos(t + \sin 100t)$$

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Tools:

- Wolfram Alpha `http://www.wolframalpha.com/`
- More powerful is Wolfram *Mathematica*. Get license from MyQC: `myqc.qc.cuny.edu/Academics/mathematics/Pages3/access.aspx`
- Online plotter: `desmos.com` Put $(f(t), g(t))$ in parentheses.

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Next time: What is the shape of a parametric curve? What is the length of a parametric curve? What about polar coordinates?

Before then: Work on homework to present in class Wednesday. Email me contact info, do syllabus quiz. Play with parametric eqns.