

Finding slope on a parametric curve

When y is a function of x , what is the slope of the tangent line?

For a parametric curve $\{x = f(t), y = g(t)\}$,

Think of y as a function of x . Then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, so $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ if _____.

- ▶ Curve has a horizontal tangent where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
- ▶ Curve has a vertical tangent where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.
- ▶ *Question:* What is true when $\frac{dx}{dt} = 0$ AND $\frac{dy}{dt} = 0$?

We can use the chain rule again to find $\frac{d^2y}{dx^2}$, but be careful!

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \underline{\hspace{2cm}}. \quad \left(\frac{dy}{dx} \text{ is a function of } \underline{\hspace{2cm}}. \right)$$

Important: $\frac{d^2y}{dx^2} \neq \frac{d^2y}{dt} / \frac{d^2x}{dt} \quad \text{!!!!!!}$

Slope of tangent line

Example. What is the tangent line to the curve $\begin{cases} x(t) = t^2 \\ y(t) = t^3 - 3t \end{cases}$ at $(3,0)$?

Question: What is t there?

Question: What is the slope there?

Question: So what is the tangent line there?

Sketching the curve

$$\begin{cases} x(t) = t^2 \\ y(t) = t^3 - 3t \end{cases}$$

Let's now sketch the curve.

Question: Where are there horizontal and vertical tangents?

▶ Horizontal:

▶ Vertical:

(Must check?)

Question: Where is the curve concave up? concave down?

▶ Calculate

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2} + \frac{3}{2t^2}}{2t} = \frac{\frac{3}{2} \frac{1}{t} \left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}.$$

Put it all together:

| t | x | y |
|-----|-----|-----|
| -3 | | |
| -1 | | |
| 0 | | |
| 1 | | |
| 3 | | |

Polar coordinates

Polar coordinates are an alternate way to think about points in 2D.

Conversions:

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \iff \begin{aligned} r^2 &= x^2 + y^2 \\ \tan \theta &= \frac{y}{x} \end{aligned}$$

| | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|-----|---|---------|---------|---------|---------|
| sin | | | | | |
| cos | | | | | |
| tan | | | | | |

Need to know

Changing coordinates:

$$(r, \theta) = (2, -\frac{2\pi}{3}) \text{ then } (x, y) =$$

$$(x, y) = (-1, 1) \text{ then } (r, \theta) =$$

Identifying polar equations:

$$\theta = 1 \qquad r = 2$$

$$r = 2 \cos \theta$$

$$r = \cos 2\theta \qquad r = 1 + \sin \theta$$

Using your calculator:

Switch to Polar mode:

MODE ↓ ↓ ↓ POL (Enter).

Also: [desmos.com](https://www.desmos.com) or Mathematica

Tangents to polar curves

Given a polar curve $r = f(\theta)$, we want to know $\frac{dy}{dx}$.

Just as before, think of y as a function of x . Then $\frac{dy}{d\theta} = \frac{dy}{dx} \cdot \frac{dx}{d\theta}$,

We conclude: $\frac{dy}{dx} = \frac{\frac{d}{d\theta}y}{\frac{d}{d\theta}x} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{(r \cos \theta + \frac{dr}{d\theta} \sin \theta)}{(r \sin \theta + \frac{dr}{d\theta} \cos \theta)}$ if _____.

Example. Find the slope of the tangent line to the curve $r = 2 \sin \theta$ at cartesian coordinates $(x, y) = (2, 0)$.