

Area under a parametric curve

Given $y = f(x)$, the area under the curve from $x = a$ to $x = b$ is

$$\text{Area} = \int_{x=a \leftarrow \text{left endpoint}}^{x=b \leftarrow \text{right endpoint}} y \, dx = \int_{t=\alpha \leftarrow \text{left endpoint}}^{t=\beta \leftarrow \text{right endpoint}} g(t) f'(t) \, dt$$

Example. Find the area under one arch of the cycloid $\begin{cases} x = r(\theta - \sin \theta) \\ y = r(1 - \cos \theta) \end{cases}$
(Here, r is a constant and θ is the parameter.)

Plot it to see the shape. One arch has range $\underline{\quad} \leq \theta \leq \underline{\quad}$.

$$\begin{aligned} A &= \int y \, dx = \int r(1 - \cos \theta) r(1 - \cos \theta) \, d\theta \\ &= r^2 \int (1 - 2 \cos \theta + \cos^2 \theta) \, d\theta \end{aligned}$$

$$= 3\pi r^2$$

Area inside a polar curve

For cartesian functions $y = f(x)$, calculate area as $A = \int dA = \int y \, dx$.

What about the area “*inside a curve*” best described as a polar function $r = f(\theta)$?

- ▶ We still use $A = \int dA$.
- ▶ But the formula for dA is different.

$$A = \int_{\theta=a}^{\theta=b} \frac{1}{2} [f(\theta)]^2 \, d\theta = \int_a^b \frac{1}{2} r^2 \, d\theta$$

Polar land

How much area is swept out by a little slice?

Example. What is the area inside one loop of the four-leaved rose $r = \cos 2\theta$?

- ▶ What are the bounds on θ ?
- ▶ For which θ does the curve pass through the origin?

$$A = \int_{\theta=\underline{\quad}}^{\theta=\underline{\quad}} \frac{1}{2} [\underline{\quad}]^2 \, d\theta$$

Inside yet Outside

Example. Calculate the area inside the curve $r = 3 \sin \theta$ and outside the curve $r = \sin \theta + 1$.

First: Draw a picture!

- ▶ What are these curves?
- ▶ Where do they intersect?

$$\begin{aligned}
 \text{Now calculate: } & \int_{\theta=\underline{\quad}}^{\theta=\underline{\quad}} \left[\overset{\text{outside}}{\left(\frac{1}{2}(3 \sin \theta)^2\right)} - \overset{\text{inside}}{\left(\frac{1}{2}(\sin \theta + 1)^2\right)} \right] d\theta \\
 = & \frac{1}{2} \int_{\theta=\underline{\quad}}^{\theta=\underline{\quad}} \left[9 \sin^2 \theta - (\sin^2 \theta + 2 \sin \theta + 1) \right] d\theta \\
 = & \frac{1}{2} \int_{\theta=\underline{\quad}}^{\theta=\underline{\quad}} \left[8 \sin^2 \theta - 2 \sin \theta - 1 \right] d\theta \\
 = & \frac{1}{2} \int_{\theta=\underline{\quad}}^{\theta=\underline{\quad}} \left[4 - 4 \cos 2\theta - 2 \sin \theta - 1 \right] d\theta \\
 = & \frac{1}{2} \int_{\theta=\underline{\quad}}^{\theta=\underline{\quad}} \left[3 - 4 \cos 2\theta - 2 \sin \theta \right] d\theta \\
 = & \left[\frac{3\theta}{2} - \sin 2\theta + \cos \theta \right]_{\pi/6}^{5\pi/6} \\
 = & \frac{3}{2} \left(\frac{5\pi}{6} - \frac{\pi}{6} \right) - \left(\sin \frac{10\pi}{6} - \sin \frac{2\pi}{6} \right) + \left(\cos \frac{5\pi}{6} - \cos \frac{\pi}{6} \right) = \dots \\
 = & \pi + \sqrt{3} - \sqrt{3} = \pi
 \end{aligned}$$

Arc length of a parametric curve

To find the arc length of a parametric curve, think $L = \int dL$.

How much arc length ΔL does the curve traverse in one time unit Δt ?

$$\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(f'(t_i^*)\Delta t_i)^2 + (g'(t_i^{**})\Delta t_i)^2}$$

(there is some t_i^* and some t_i^{**} in the time interval...)

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta L_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{(f'(t_i^*))^2 + (g'(t_i^{**}))^2} \Delta t_i$$

(Similar to a Riemann Sum)

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

Example. Find the arc length for the parametric curve $\begin{cases} x = \sin 2t \\ y = \cos 2t \end{cases}$ for $0 \leq t \leq 2\pi$. **What do we expect? What is this curve?**

$$\int_{t=0}^{t=2\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt =$$

Arc length of a polar curve

To calculate arc length, view a polar curve as a parametric curve.

▶ Convert $r = f(\theta)$ to
$$\begin{cases} x = f(\theta) \cos \theta \\ y = f(\theta) \sin \theta \end{cases}$$

▶ Then
$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[\frac{d}{d\theta} (f(\theta) \cos \theta) \right]^2 + \left[\frac{d}{d\theta} (f(\theta) \sin \theta) \right]^2} d\theta$$

... take derivatives & do the algebra ...

▶
$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{[f(\theta)]^2 + \left[\frac{d}{d\theta} (f(\theta)) \right]^2} d\theta$$

We can also understand this as $dL = \sqrt{(r d\theta)^2 + (dr)^2}$