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$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

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$\int_{t=0}^{t=2\pi} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\,dt =$

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- Then $L = \displaystyle\int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[\frac{d}{d\theta}\big(f(\theta)\cos\theta\big)\right]^2 + \left[\frac{d}{d\theta}\big(f(\theta)\sin\theta\big)\right]^2}\, d\theta$

. . . take derivatives & do the algebra . . .

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Arc length of a polar curve

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We can also understand this as $dL = \sqrt{(r\, d\theta)^2 + (dr)^2}$