

Course Notes

Mathematical Models, Spring 2015

Queens College, Math 245

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<http://qcpages.qc.edu/~chanusa/courses/245/15/>

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- ▶ Model people waiting using a **computer simulation**.
- ▶ Model allocating resources using a **system of inequalities**.

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Then we must **analyze our models** to determine their applicability.

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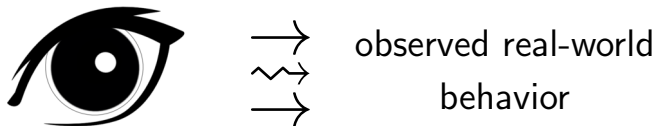


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- ▶ What is happening? (Observation)
- ▶ What are the reasons for the behavior? (Hypothesis)
- ▶ How do we convey that our reasoning is plausible? (“proof”)

— Use the language of mathematics! —

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Goal: Understand what is involved in “mathematical modeling”.

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- ▶ **Describe mathematically.** Assign each quantity a variable. Represent each relationship with an equation.

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Why do objects fall? –to– *How* do objects fall?

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 Other variables?: _____
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 And *proportional* means $v = ax$ for some constant a . (Goal?)

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We have both $v = \frac{dx}{dt}$ and $v = ax$. Set them equal.

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- ▶ Has the model explained the real-world observations?
- ▶ Are the answers we found accurate enough?
- ▶ Were our assumptions good assumptions?
- ▶ What are the strengths and weaknesses of our model?
- ▶ Did we make any mistakes in our mathematical manipulations?

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If there are any problems,

- ▶ **Go back** to the First Step, Formulation.
- ▶ Change your assumptions!
- ▶ Start the modeling process over.

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Perhaps the proportionality assumption is incorrect?

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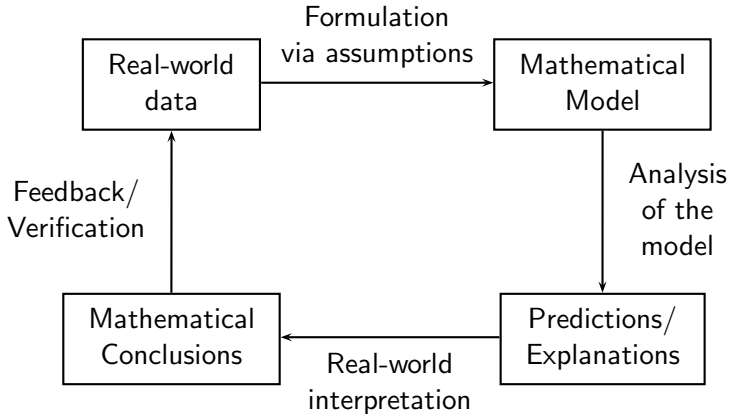
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(Although not all!)

The Modeling Process

This chart summarizes the modeling process.



To do well in this class:

- ▶ **Come to class prepared.**
 - ▶ Print out and read over course notes.
 - ▶ Read assigned sections before class.
- ▶ **Form good study groups.**
 - ▶ Discuss homework and classwork.
 - ▶ Final project is a group project.
 - ▶ You will depend on this group.
- ▶ **Put in the time.**
 - ▶ Three credits = (at least) nine hours / week out of class.
 - ▶ Homework stresses key concepts from class; learning takes time.
- ▶ **Stay in contact.**
 - ▶ If you are confused, ask questions (in class and out).
 - ▶ Don't fall behind in coursework or project.
 - ▶ I need to understand your concerns.

Homework posted online; Email me by Monday.

Choosing a problem statement.

Group Activity. Arrange yourselves into groups of four or five people, with people you **don't know**.

- ▶ Introduce yourself. (your name, where you're from, your major)
- ▶ Fill out **the front of** your notecard:
 - ▶ Write your name. (Stylize if you wish.)
 - ▶ Write a few words related to your name.
 - ▶ *Draw* something in the remaining space.
- ▶ Discuss with your groupmates why you wrote what you wrote.
- ▶ Exchange contact information. (phone / email / other)

- ▶ Work in your group on the worksheet.