

# Vectors

We will be using vectors and matrices to store and manipulate data.

*Definition:* A **vector**  $\vec{v}$  is a column of numbers. Use bold faced letters or vector signs to distinguish vectors from other variables.

We refer to the **entries** of a vector by using subscripts.

The **length** of a vector is the number of entries it has. (normally  $n$ )

*Example.*  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1 \ 2 \ 3]^T.$

*Example.* Use a vector to represent the age distribution of a population: let  $F_i$  be the number of females with ages in the interval  $[5i, 5(i + 1))$ . We can represent the total female population by the vector  $\vec{F}$ .

The females from 0 up to 5 are counted in  $F_0$ ; those from 5 up to 10 are counted in  $F_1$ , etc.

$$\vec{F} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{bmatrix}$$

# Matrices

*Definition:* A **matrix**  $A$  is a two-dimensional array of numbers.

A matrix with  $m$  rows and  $n$  columns is called an  $m \times n$  matrix.

★ Row by column — Row by column — Row by column ★

*Note:* A vector can be thought of as an  $n \times 1$  matrix.

Matrices are denoted by a capital letter. Entries are lower case and have two subscripts, the corresponding row and column.

*Example.* A generic  $2 \times 3$  matrix has the form  $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$ .

*Definition:* The matrix  $B = \begin{bmatrix} 30 & 50 \\ 100 & 250 \end{bmatrix}$  is a **square matrix** because it has the same number of rows as columns.

# Matrices

**Example.** We will sometimes interpret a matrix as a **transition** matrix. In this case, the matrix is square (say  $n \times n$ ), where the  $n$  rows and  $n$  columns correspond to certain **states** (situations).

An entry  $a_{i,j}$  represents transitioning from state  $j$  to state  $i$ .

**Example.** In our population example, suppose we want to model people getting older, transitioning from one state (age group) to the next. We would set up a transition matrix such as:

<b>TO state:</b>	<b>FROM state:</b>	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$	,	<p>because everyone in the first age group will move to the second age group (<math>a_{2,1}</math>), everyone in state 2 will move to state 3 (<math>a_{3,2}</math>), etc.</p>
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# Matrix Multiplication

The power of matrices arises in their multiplication.

Given two matrices,  $A$  of size  $m \times k$  and  $B$  of size  $l \times n$ , we can find the product  $AB$  **if and only if**  $k$  equals  $l$ .

Let  $A$  be an  $m \times k$  matrix and  $B$ ,  $k \times n$ . Then  $AB$  is of size  $m \times n$ .

To calculate the entries of  $AB$ , remember: “Row by column”:

$$\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 6 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix}$$

When we write  $A^2$ , this means  $AA$ ;  $A^3$  means  $AAA$ , etc.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \circ & \circ \\ 0 & 1 & \circ \\ 0 & 0 & 1 \end{bmatrix}$$

# The power of transition matrices

**Example.** Modeling a changing population using a matrix model.

Let us choose a size of age interval  $\Delta=5$  years (“Delta”), and divide the female population into states:

*age distribution vector:*

State 0: ages [0, 5) with $F_0 = 150$ females	$\vec{F} = \begin{bmatrix} 150 \\ 200 \\ 180 \\ 120 \\ 60 \end{bmatrix}$
State 1: ages [5, 10) with $F_1 = 200$ females	
State 2: ages [10, 15) with $F_2 = 180$ females	
State 3: ages [15, 20) with $F_3 = 120$ females	
State 4: ages [20, 25) with $F_4 = 60$ females	

Using a transition matrix, we can determine the population in 5 years:

$$A \cdot \vec{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^1 \begin{bmatrix} 150 \\ 200 \\ 180 \\ 120 \\ 60 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \\ 200 \\ 180 \\ 120 \end{bmatrix}$$

# Leslie Matrices

The transition matrix in the previous example is not entirely realistic, because people die and are born

To take death into account, modify:

To take birth into account, modify:

( $j$  females !)

The resulting transition matrix is called a **Leslie matrix**:

Let  $m_i$  be the average number of females that women in state  $i$  bear.

Let  $p_i$  be the fraction of women in state  $i$  that survive to state  $i + 1$ .

$$\text{then } \begin{bmatrix} F_0(t + \Delta) \\ F_1(t + \Delta) \\ F_2(t + \Delta) \\ \vdots \\ F_{n-1}(t + \Delta) \end{bmatrix} = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ p_0 & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & p_{n-2} & 0 \end{bmatrix} \begin{bmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_{n-1}(t) \end{bmatrix}$$

$$\vec{F}(t + \Delta) = M \cdot \vec{F}(t)$$

# Leslie Matrices

**Example.** An animal population example (p. 47)

The population in three age groups,  $F_0 = 80$ ,  $F_1 = 40$ , and  $F_2 = 20$ .

Suppose that as  $\Delta$  time passes, everyone in state 2 dies, and one quarter of everyone else dies. Also suppose that the age-specific maternity rates are  $m_0 = 0$ ,  $m_1 = 1$ , and  $m_2 = 2$ . Determine the Leslie matrix and the population distributions at times  $\Delta$  and  $2\Delta$ .

$$\begin{bmatrix} & & \\ & 0 & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} = \vec{\mathbf{F}}(\Delta)$$

$$\begin{bmatrix} & & \\ & 0 & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} = \vec{\mathbf{F}}(2\Delta)$$

# Matrices and Vectors in Mathematica

We can use Mathematica to do these calculations. (See Tutorial 5.)

- ▶ Matrices are defined as lists of row vectors.

```
matrix = {{0, 1, 2}, {.75, 0, 0}, {0, .75, 0}}  
vector = {{80}, {40}, {20}}
```

- ▶ Multiply matrices by using a period (.)

```
matrix.vector
```

- ▶ Find powers of matrices using `MatrixPower`, not `^`  
`MatrixPower[matrix, 2]`

- ▶ So, to find animal population over time, use the code:  
`Table[  
 , {i, 0, 10}]`