Markov Chains

A **Markov chain** is a sequence of random variables from one sample space, each corresponding to a successive time interval. From one time interval to the next, there is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i. No transition depends on a past transition.

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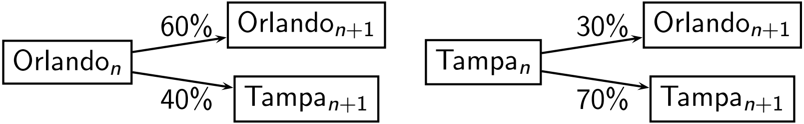
Example. Suppose you run a rental company based in Orlando and Tampa, Florida. People often drive between the cities; cars can be picked up and dropped off in either city. Suppose that historically,



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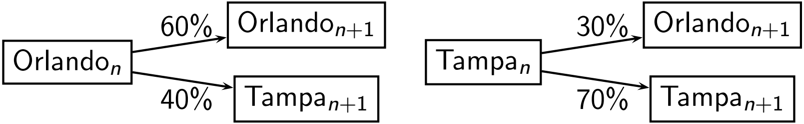

What distribution of cars can the company expect in the long run?

Keep track of these probabilities in an associated transition matrix A.

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

$$\begin{array}{cc} & \textbf{FROM}: \\ & \begin{array}{cc}\text{Or} & \text{Tm}\end{array} \\ \textbf{TO}: \begin{array}{c}\text{Or} \\ \text{Tm}\end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = A, \end{array}$$

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Given an initial distribution $\vec{\mathbf{x}}_0 = \begin{bmatrix} o_0 \\ t_0 \end{bmatrix}$,

the expected distribution of cars at time n is $\vec{\mathbf{x}}_n =$ ____.

Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

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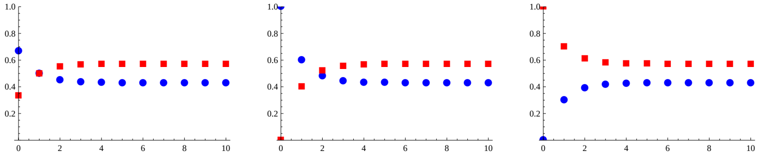

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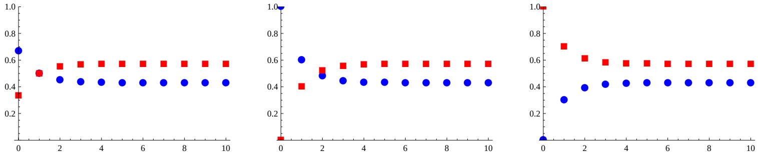

How do we determine the expected distribution in the long run?

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So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
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★ There is no general rule for what the row sum will be.

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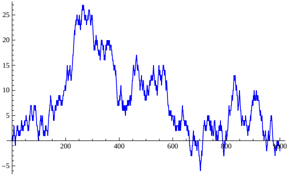

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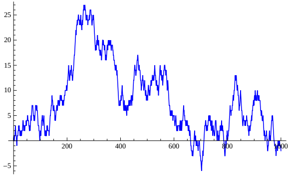
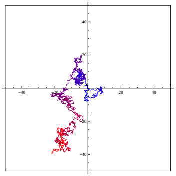


Movement of a molecule in liquid (Wiener process)



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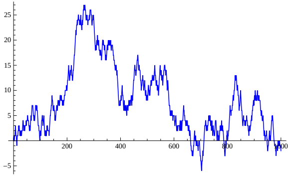

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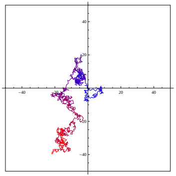

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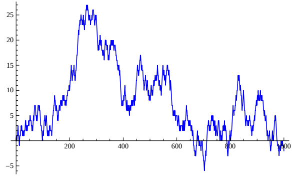

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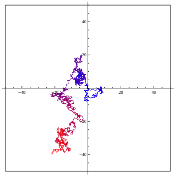

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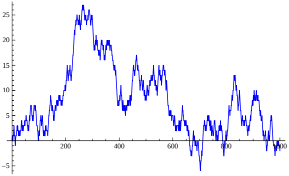

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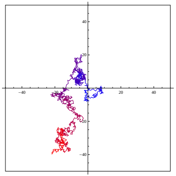

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We transition from one state to another by some rule. Perhaps:

- Moving a random card to a new position.
- Choosing a pair of random cards and exchanging them.

Simple random walk

A drunk in a bar. A bar patron has had a little too much to drink and it's about time to leave the bar. There is an exit directly to his right and an exit three steps away to his left. The drunk stumbles randomly one step to the left or one step to the right with equal probability.

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What is an equilibrium solution for this random walk?

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Win or go home broke! A gambler starts with \$500 and makes \$1 bets, winning each with probability p.
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There also exist higher-dimensional random walks.

Color mixing game

Let's play an interactive Markov chain game.

- Choose a color. (Red, Orange, Yellow, Green, Blue, Purple)
- Record the distribution.

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- Record the distribution at multiple times during the experiment.

What do we expect to occur?

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What do we expect to occur?

Stand up and make some space to move around.