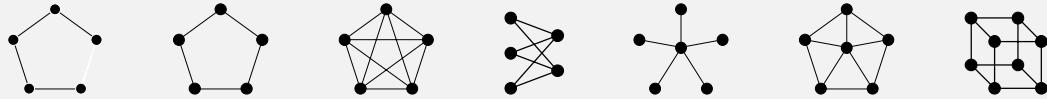


Families of Graphs



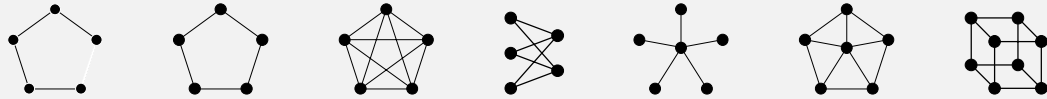
- ▶ **Path graph P_n** : The path graph P_n has $n + 1$ vertices, $V = \{v_0, v_1, \dots, v_n\}$ and n edges, $E = \{v_0v_1, v_1v_2, \dots, v_{n-1}v_n\}$.
 - ★ The **length** of a path is the number of *edges* in the path.

- ▶ **Cycle graph C_n** : The cycle graph C_n has n vertices $V = \{v_1, \dots, v_n\}$ and n edges, $E = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$.

We often try to find and/or count paths and cycles in a graph.

Question. What is the shortest path? Largest cycle?

Families of Graphs

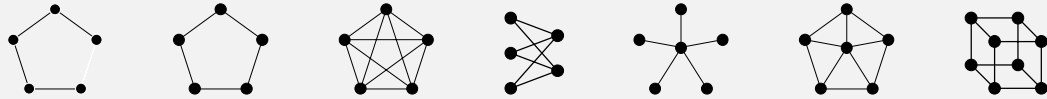


- ▶ **Complete graph K_n** : The complete graph K_n has n vertices, $V = \{v_1, \dots, v_n\}$ and has an edge connecting every pair of distinct vertices, for a total of _____ edges.

Definition. A **bipartite** graph is a graph where the vertex set can be broken into two parts such that there are no edges between vertices in the same part.

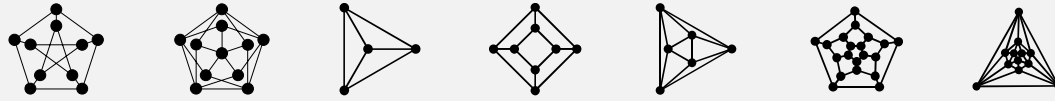
- ▶ **Complete bipartite graph $K_{m,n}$** : The complete bipartite graph $K_{m,n}$ has $m + n$ vertices $V = \{v_1, \dots, v_m, w_1, \dots, w_n\}$ and an edge connecting each v vertex to each w vertex.

Families of Graphs



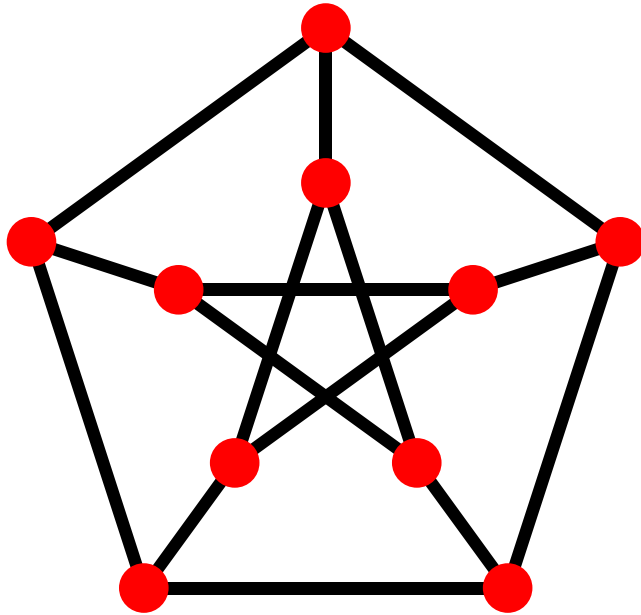
- ▶ **Star graph** St_n : The star graph St_n has $n + 1$ vertices $V = \{v_0, v_1, \dots, v_n\}$ and n edges $E = \{v_0v_1, v_0v_2, \dots, v_0v_n\}$.
- ▶ **Wheel graph** W_n : The wheel graph W_n has $n + 1$ vertices $V = \{v_0, v_1, \dots, v_n\}$. Arrange and connect $\{v_1, \dots, v_n\}$ in a cycle (the rim of the wheel). Connect v_0 to every other vertex. (We call v_0 the hub.)
- ▶ **Cube graph** \square_n : The cube graph in n dimensions, \square_n , has 2^n vertices. We index the vertices by binary numbers of length n . Two vertices are adjacent when their binary numbers differ by exactly one digit.

Special Graphs

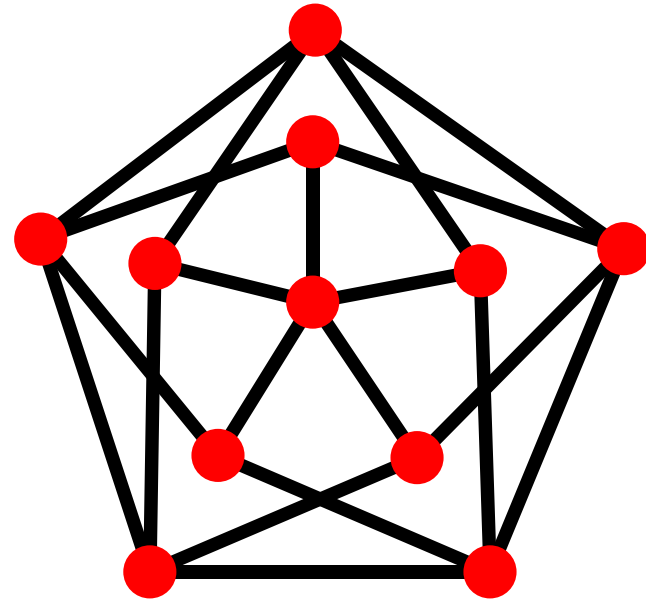


Two graphs we will see on a consistent basis are:

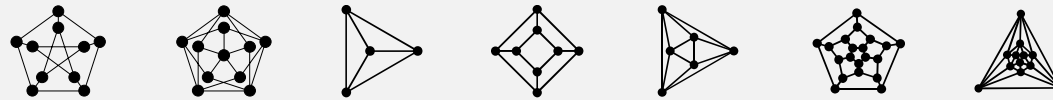
Petersen graph P



Grötzsch graph G_r



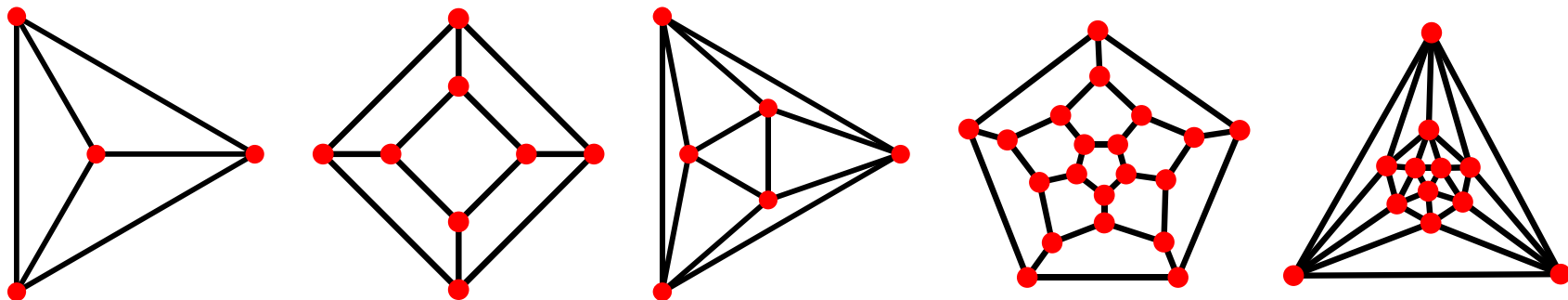
Special Graphs



Definition. The **platonic solids** are the tetrahedron, cube, octahedron, icosahedron, and dodecahedron. They are the only regular convex polyhedra made of regular polygons.

Definition. The **Schlegel diagram** of a polyhedron is a planar 2D graph that represents a 3D object, where vertices of the graph represent vertices of the polyhedron, and edges of the graph represent the edges of the polyhedron.

- ▶ The **Platonic graphs** are the Schlegel diagrams of the five platonic solids.



- ▶ What do you notice? What do you wonder?


When are two graphs the same?

Two graphs G_1 and G_2 are **equal** ($G_1 = G_2$) if they have the **exact same** vertex sets and edge sets.

The graphs G_1 and G_2 are **isomorphic** ($G_1 \approx G_2$) if there exists a **bijection** on the vertex sets, $\varphi : V(G_1) \rightarrow V(G_2)$ such that

$$v_i v_j \text{ is an edge of } G_1 \quad \text{iff} \quad \varphi(v_i)\varphi(v_j) \text{ is an edge of } G_2.$$

In this course, we will spend a large amount of time trying to figure out whether two given graphs are isomorphic.

Side note: The homomorphisms of a graph (isomorphisms into itself) is a measure of its symmetry. *Example.* 

Simple operations on graphs

The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ can mean two different things:

- ▶ When the vertex sets are different, the **(disjoint) union** H of G_1 and G_2 is formed by placing the graphs side by side. In this case, $H = (V_1 \cup V_2, E_1 \cup E_2)$.
- ▶ When the vertex sets are the same, then the **(edge) union** H of G_1 and G_2 contains every edge of both E_1 and E_2 . In this case, $H = (V, E_1 \cup E_2)$.

The **complement** G^c or \overline{G} of a graph $G = (V, E)$ is a graph with vertex set V and whose edge set contains all edges **NOT** in G .

Consequence: Suppose $G = (V, E_1)$ and $G^c = (V, E_2)$. Then $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E(K_{|V|})$. (Recall K_n : complete graph.)

Subgraphs

A **subgraph** H of a graph G is a graph where every vertex of H is a vertex of G , and where every edge of H is an edge of G .

★ If edge e of G is in H , then the endpoints of e must also be in H .

A subgraph H is a **proper subgraph** if $H \neq G$.

If G_1 and G_2 are two graphs, we say that G_1 **contains** G_2 if there exists a subgraph H of G_1 such that H is isomorphic to G_2 .

Example. Show that W_6 contains C_3 , C_4 , C_5 , C_6 , and C_7 .

Induced Subgraphs

For a graph $G = (V, E)$ and any subset $W \subseteq V(G)$, we can define the subgraph of G **induced by W** .

Define H :

- ▶ $V(H) = W$
- ▶ $E(H) =$ edges in $E(G)$ that have endpoints *exclusively in W* .

Any graph that could be defined in this way is called an **induced subgraph** of G .

Induced subgraphs of G are always subgraphs of G , but not vice versa.