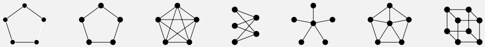
# Families of Graphs



- **Path graph** $P_n$: The path graph $P_n$ has $n+1$ vertices, $V = \{v_0, v_1, \ldots, v_n\}$ and $n$ edges, $E = \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n\}$.
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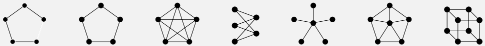

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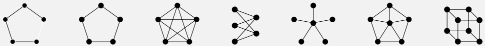

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We often try to find and/or count paths and cycles in a graph.

*Question.* What is the shortest path? Largest cycle?

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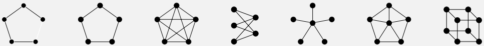

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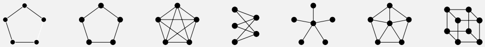

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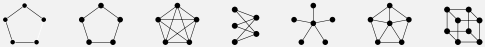

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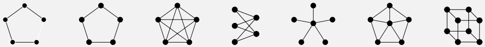

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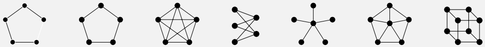

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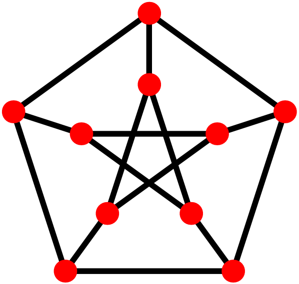
- **Cube graph** $\square_n$: The cube graph in $n$ dimensions, $\square_n$, has $2^n$ vertices. We index the vertices by binary numbers of length $n$. Two vertices are adjacent when their binary numbers differ by exactly one digit.

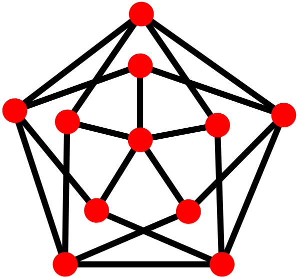
# Special Graphs

Two graphs we will see on a consistant basis are:

**Petersen graph** $P$



**Grötzsch graph** $Gr$



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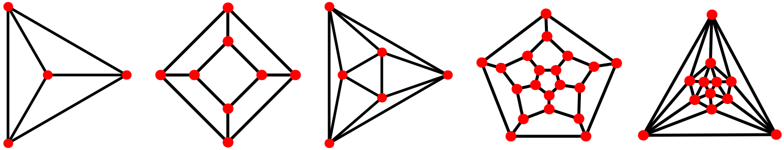

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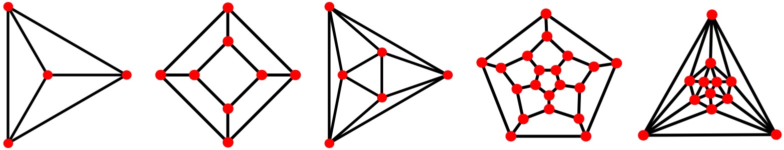

- What do you notice? What do you wonder?

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Side note: The homomorphisms of a graph (isomorphisms into itself) is a measure of its symmetry. *Example.* 

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*Consequence:* Suppose $G = (V, E_1)$ and $G^c = (V, E_2)$. Then $E_1 \cap E_2 = \varnothing$ and $E_1 \cup E_2 = E(K_{|V|})$. (Recall $K_n$: complete graph.)

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*Example.* Show that $W_6$ contains $C_3$, $C_4$, $C_5$, $C_6$, and $C_7$.

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Induced subgraphs of $G$ are always subgraphs of $G$, but not vice versa.

# Questions for deep thought:

Determine the complement of the graph $K_{m,n}$ if $m, n >= 3$.
Prove or disprove: Every subgraph of the complete graph $K_n$, $n \geq 3$, contains a cycle.
Prove or disprove: Every induced subgraph of a wheel graph $W_n$, $n \geq 3$, contains a cycle
Prove or disprove: Any induced subgraph of a complete graph is a complete graph.
Prove or disprove: Any induced subgraph of a complete bipartite graph is a complete bipartite graph. Prove that the Schlegel diagram of the cube is isomorphic to the Cube graph $\square_3$.
Is the wheel graph regular for any value of $n$? Cycle graph? the cube graph?