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- If not all vertices are distinct, then choose the *first* vertex v_p in P that is also a vertex w_q in Q.

Lemmas A and B

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Write the cycle C as $C = x z_1 z_2 \cdots z_k y x$.
Therefore, there is a path $Q : x \to y = x z_1 z_2 \cdots z_k y$ in H.

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Claim: There is a path from v to w in H. Why?

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Break into cases, depending on whether G contains a cycle:

(next page)

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★ Important Induction Item: Always **remove** edges. ★

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Trees are the smallest connected graphs; the following theorems show this and help classify graphs that are trees.

Thm 1.3.2, 1.3.3: Let G be a connected graph with p vertices and q edges. Then,

$$G \text{ is a tree} \iff p = q + 1.$$

Thm 1.3.5. G is a tree iff there exists exactly one path between each pair of vertices.

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(b) For v_1, v_2 in a cycle, there exist two paths between v_1 and v_2.

In both cases, it is not the case that between each pair of vertices, there exists exactly one path.

Related theorems

Definition. A **bridge** is an edge e such that its removal disconnects G.

Theorem 2.4.1. Suppose that G is a connected. Then

$$G \text{ is a tree} \iff \text{Every edge of } G \text{ is a bridge.}$$

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Proof. Let G be a regular graph of even degree with a bridge $e = vw$.