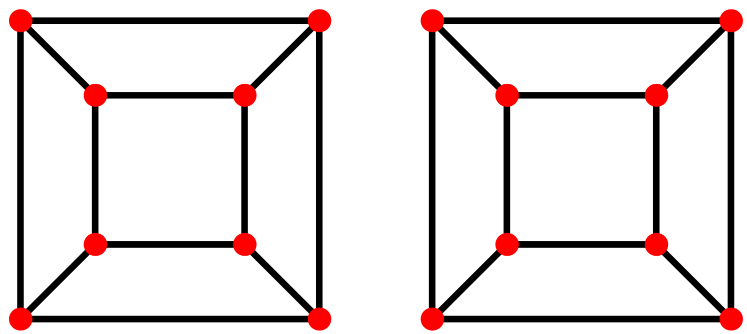
Edge Coloring

We can also color the edges of a graph.

Definition. An **edge coloring** of a graph G is a labeling of the edges of G with colors. [Technically, a function $f : E(G) \to \{1, 2, \ldots, l\}$.]

Definition. A **proper** edge coloring of G is an edge coloring of G such that no two *adjacent edges* are colored the same.

Example. Cube graph ($\square_3$):



We can properly edge color $\square_3$ with ____ colors and no fewer.

Definition. The minimum number of colors necessary to properly edge color a graph G is called the **edge chromatic number** of G, denoted $\chi'(G) =$ “chi prime”.

Edge coloring theorems

Question. What is a natural lower bound for $\chi'(G)$?

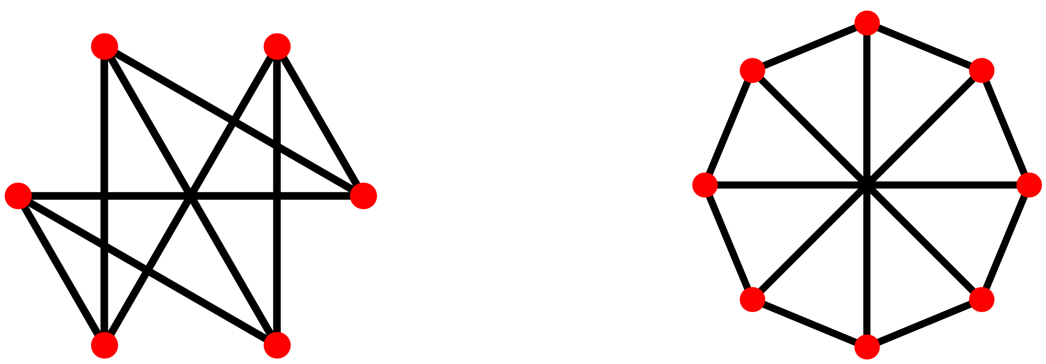
Thm 2.2.1: For any graph G, $\chi'(G) \geq$ _____.

Thm 2.2.2: Vizing's Theorem:
For every graph G, $\chi'(G)$ equals either $\Delta(G)$ or $\Delta(G) + 1$.

Proof. Hard. (See reference [24] if interested.)

Consequence: To determine $\chi'(G)$,

Fact: **Most** 3-regular graphs have edge chromatic number 3.

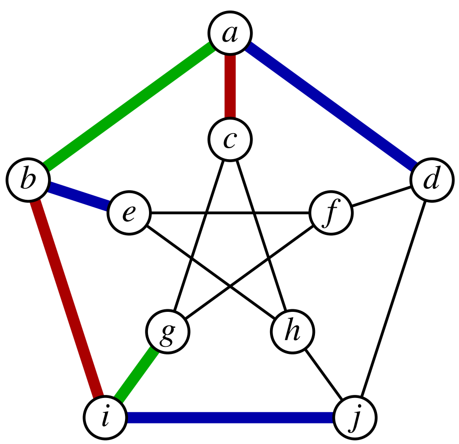


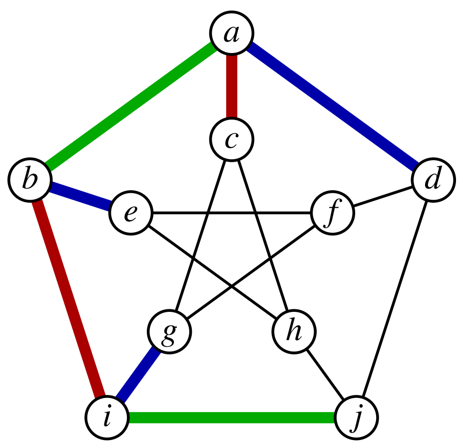
Snarks

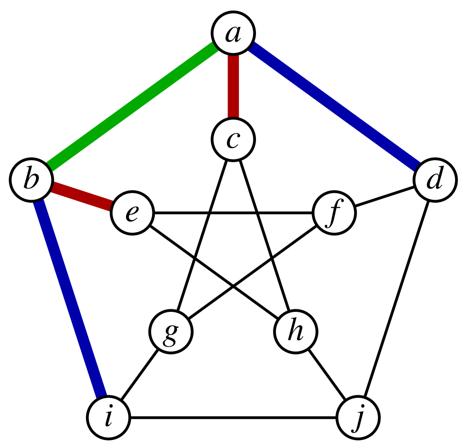
Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij.







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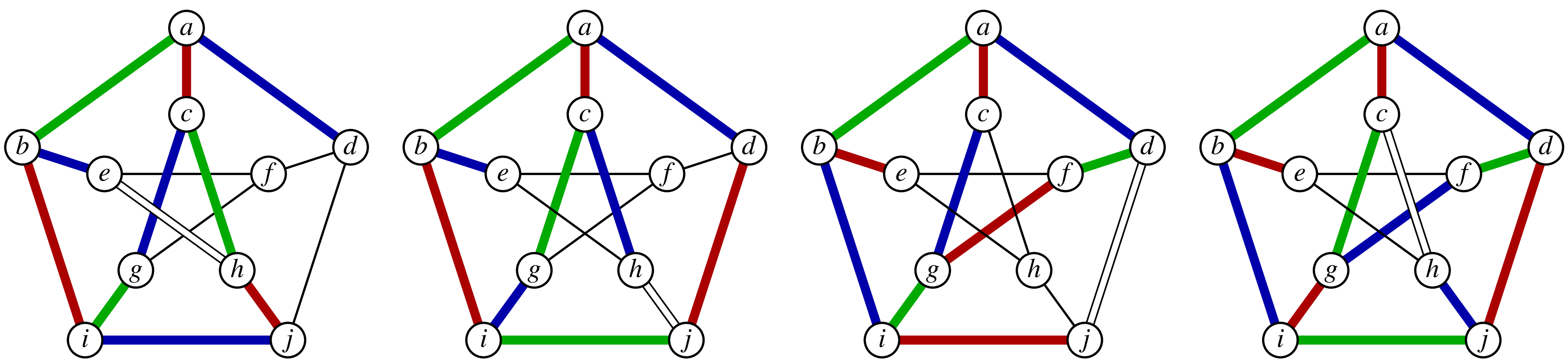
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij. **Cases 2a, 2b**



In all cases, it is not possible to edge color with 3 colors, so $\chi'(G) = 4$.

The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

Vertex Degree Analysis: The degree of every vertex in K_n is ____.

Vizing's theorem implies that $\chi'(K_n) =$ ____ or ____.

If $\chi'(K_n) =$ ____, then each vertex has an edge leaving of each color.

Question. How many red edges are there?

This is only an integer when:

So, the best we can expect is that $\begin{cases} \chi'(K_{2n}) = \\ \chi'(K_{2n-1}) = \end{cases}$

The edge chromatic number of complete graphs

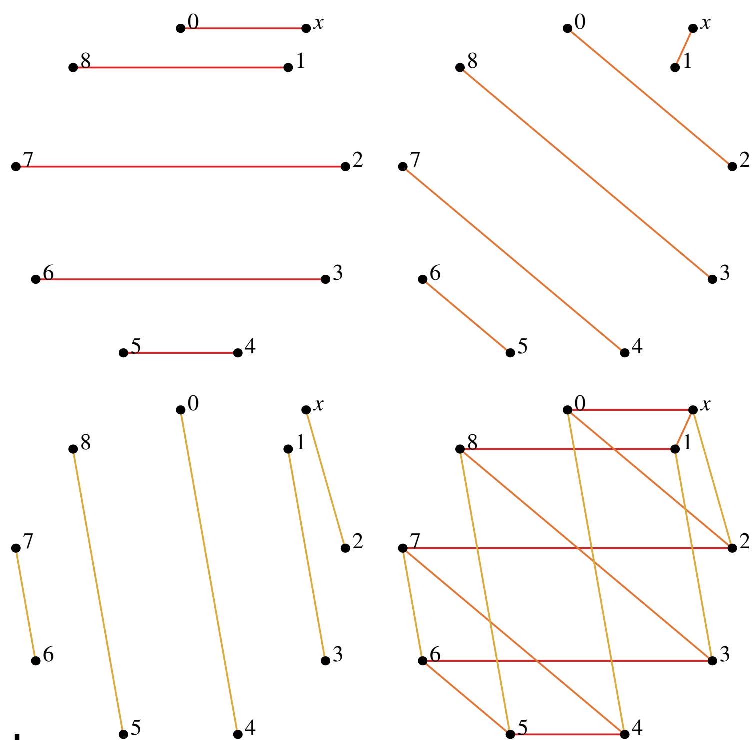
Thm 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

Proof. Use the *"turning trick"*.

Label the vertices of K_{2n}
$0, 1, \ldots, 2n-2, x$.

Connect 0 with x,
Connect 1 with $2n-2$,
$\vdots$
Connect $n-1$ with n.

Now **turn** the inside edges.
And do it again. (and again, . . .)



Claim: Each turn, new edges are used.

Proof: Each of the edges is a different "circular length".
Vertices are at circular distance 1, 3, 5, . . ., 4, 2 from each other, and x is connected to a different vertex each time.

The edge chromatic number of complete graphs

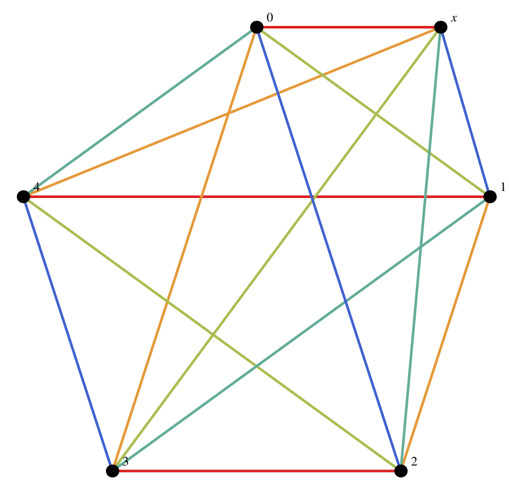
Theorem 2.2.4: $\chi'(K_{2n-1}) = 2n - 1$.

This construction also gives a way to edge color K_{2n-1} with $2n-1$ colors—simply delete vertex x!

This is related to the mathematics of **combinatorial designs.**

Question. Is it possible for six tennis players to play one match per day in a five-day tournament in such a way that each player plays each other player once?

Day 1	0x	14	23
Day 2	1x	20	34
Day 3	2x	31	40
Day 4	3x	42	01
Day 5	4x	03	12



Theorem 2.2.3 proves there is such a tournament for all even numbers.