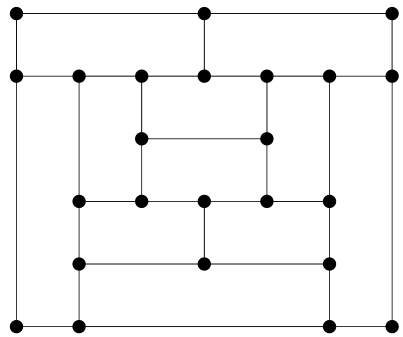
Hamiltonian Cycles

Definition. A **Hamiltonian cycle** C in a graph G is a cycle containing every vertex of G.



Definition. A **Hamiltonian path** P in a graph G is a path containing every vertex of G.

★ Important: Paths and cycles do not use any vertex or edge twice. ★

Theorem: If G has a Ham'n cycle, then G has a Ham'n path.
Proof:

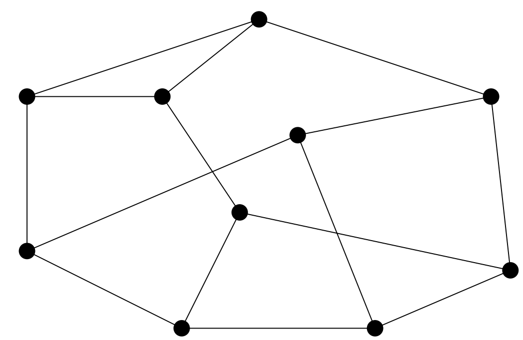
An arbitrary graph may or may not contain a Hamiltonian cycle/path. This is very hard to determine in general!

Hamiltonian Cycles

Theorem 2.3.5: A snark has no Hamiltonian cycle.

Fact: A snark has an even number of vertices. Why?

Proof: Suppose that G is a snark that contains a Hamiltonian cycle C, visiting each vertex once.

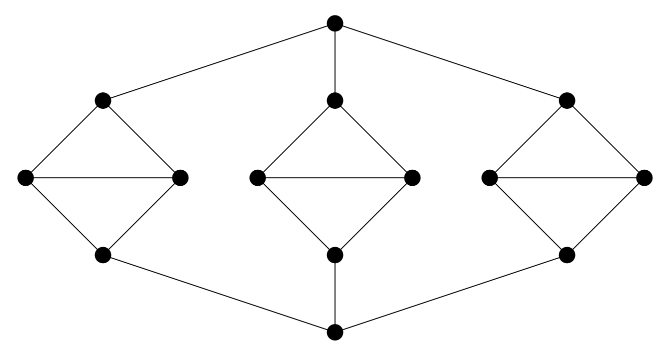


When you remove C . . .

Now color G strategically . . .

Careful: The converse is not true!

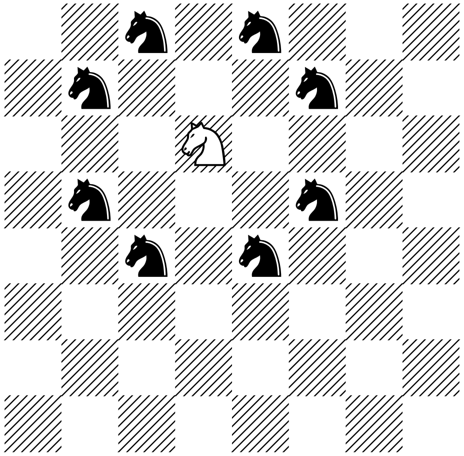
There exist cubic graphs w/o Ham'n cycle and that are not snarks.



Example: Book Figure 2.3.4.

Knight's Tours

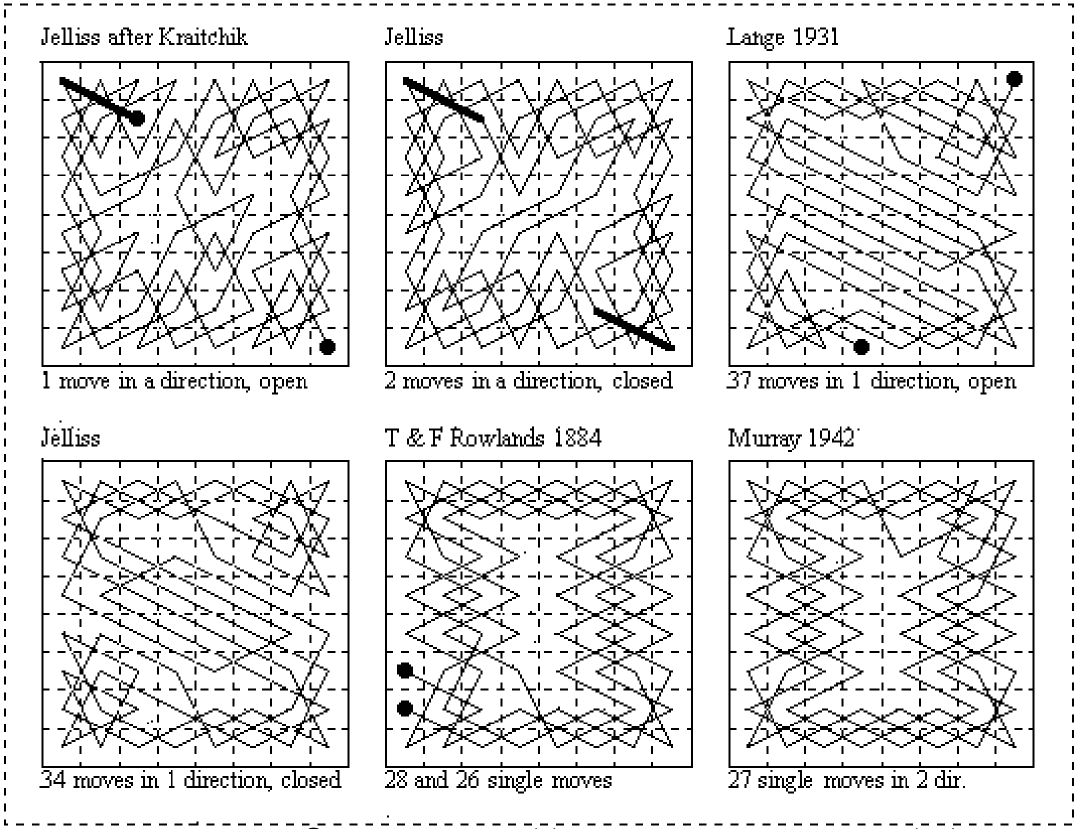
In chess, a knight (♘) is a piece that moves in an "L": two spaces over and one space to the side.



Question. Is it possible for a knight to start on some square and, by a series of valid knight moves, visit each square on an 8×8 chessboard once? (How about return to where it started?)

Definition. A path of the first kind is called an **open knight's tour**. A cycle of the second kind is called a **closed knight's tour**.

8×8 Knight's Tour

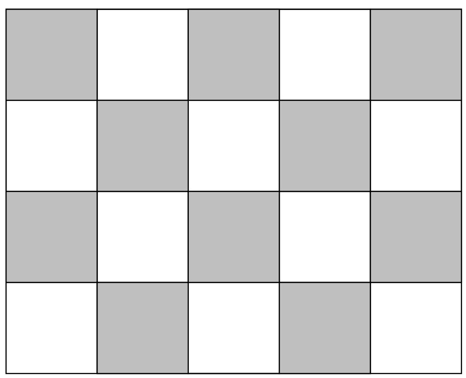


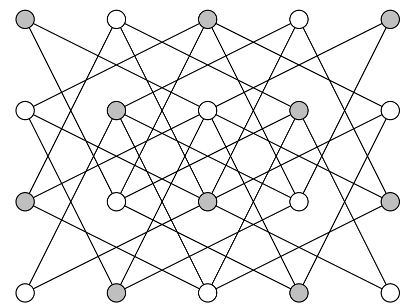
Source: `http://www.mayhematics.com/t/8g.htm`

Question. Are there any knight's tours on an $m \times n$ chessboard?

The Graph Theory of Knight's Tours

For any board we can draw a corresponding knight move graph: Create a vertex for every square on the board and create edges between vertices that are a knight's move away.





An open/closed knight's $\longleftrightarrow$
tour on the board

A knight move always alternates between white and black squares. Therefore, a knight move graph is always ________________.

Knight's Tour Theorem

Theorem. An $m \times n$ chessboard with $m \leq n$ has a *closed* knight's tour unless one or more of these conditions holds:

1. m and n are both odd.
2. $m = 1$, 2, or 4.
3. $m = 3$ and $n = 4$, 6, or 8.

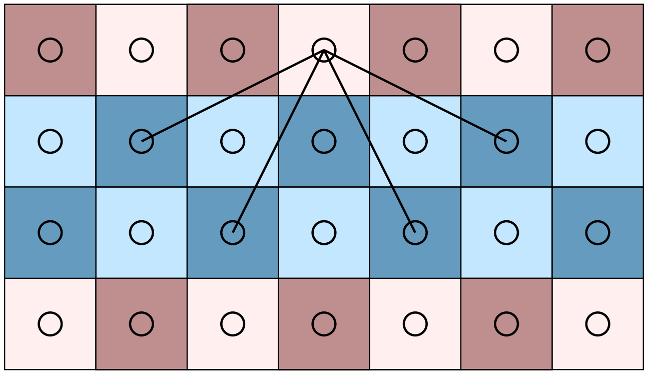
"Proof" We will only show that it is impossible in these cases.

Case 1. When m and n are both odd,

Case 2. When $m = 1$ or 2, the knight move graph is not connected.

Knight's Tour Theorem

Case 2. When $m = 4$, draw the knight move graph G.



Suppose there exists a Hamiltonian cycle C in the graph G. Since G is bipartite, C must alternate between white and black vertices.

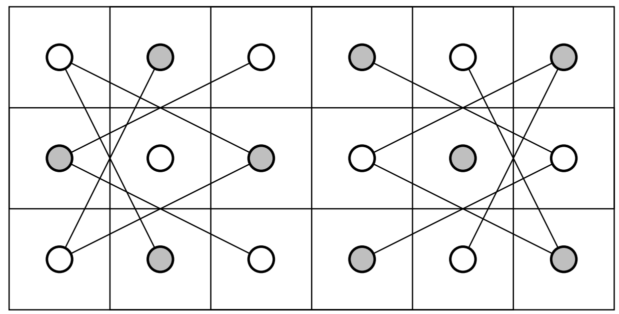
In addition, tint the outer rows of G red and the inner rows blue. **In** C**,** every red vertex is only adjacent to blue vertices.

Since there are the same number of red and blue vertices, C must also alternate between red and blue vertices.

Therefore: All vertices of C are "white and red" or "black and blue".

Knight's Tour Theorem

Case 3. 3×4 is covered by Case 2. Consider the 3×6 board:



Assume that there is a Hamiltonian cycle C in G.

C visits every vertex v and uses two of v's incident edges.

If $\deg_G(v) = 2$, then both of v's incident edges in G are in C.

Draw all these "forced edges" that must be in C.

The forced edges include four edges that form a cycle C'.

This cycle C' cannot be a subgraph of any Hamiltonian cycle! $\Rightarrow\!\!\times\!\!\Leftarrow$

The 3×8 case is similar, and for you to explore.

See also: "Knight's Tours on a Torus", by J. J. Watkins, R. L. Hoenigman