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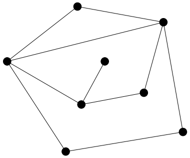
Example. K_4 is planar because there exists a plane drawing of K_4.

Vertices, Edges, and Faces

Definition. In a plane drawing, edges divide the plane into **regions**, or **faces**.



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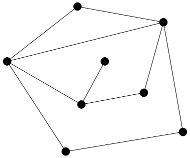

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Notation. Let $p = \#$ of vertices, $q = \#$ of edges, $r = \#$ of regions. Compute the following data:

Graph	p	q	r	
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

In 1750, Euler noticed that ____________ in each of these examples.

Euler's Formula

Theorem 8.1.1 (Euler's Formula) If G is a connected planar graph, then in a plane drawing of G, $p - q + r = 2$.

Proof. (by induction on the number of cycles)

Base Case: If G is a connected graph with no cycles, then G _____

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Now remove e: Define $H = G \setminus e$. Now H has fewer cycles than G, and one fewer region. The inductive hypothesis holds for H, giving:

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What do we notice about these graphs?

Numerical Conditions on Planar Graphs

Claim. Every face of a maximal planar graph is a ____________!

Proof. Otherwise,

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Question. Do we need $p \geq 3$?

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Corollary 8.1.3. Every planar graph with $p \geq 3$ vertices has at most $3p - 6$ edges.

- Start with any planar graph G with p vertices and q edges.
- Add edges to G until it is maximal planar. (with $Q \geq q$ edges.)
- This resulting graph satisfies $Q = 3p - 6$; hence $q \leq 3p - 6$.

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Theorem 8.1.4. The graph K_5 is not planar.

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Theorem 8.1.7. Every planar graph has a vertex with degree ≤ 5.

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Recall: The **girth** $g(G)$ of a graph G is the smallest cycle size.

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Proof. Modify the above proof—instead of $3r = 2q$, we know $4r \leq 2q$. This implies that

$$2 = p - q + r \leq p - q + \frac{2q}{4} = p - \frac{q}{2}.$$

Therefore, $q \leq 2p - 4$.

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Theorem 8.1.5. If G is planar and bipartite, then $q \leq 2p - 4$.

Theorem 8.1.6. $K_{3,3}$ is not planar.