

Modifications of Graphs

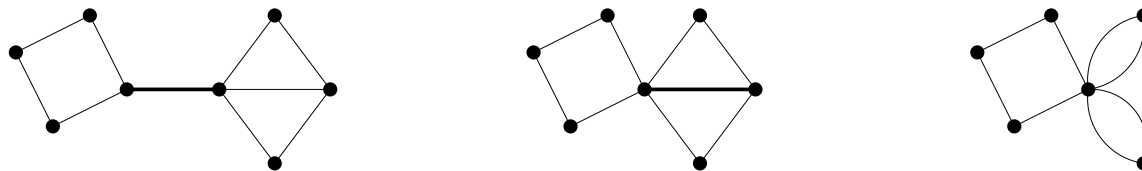
Definition. Deletion

$G \setminus v$ (G delete v): Remove v from the graph and all incident edges.

$G \setminus e$ (G delete e): Remove e from the graph.

Definition. Contraction

G/e (G contract e): If $e = vw$, coalesce v and w into a super-vertex adjacent to all neighbors of v and w . [*This may produce a multigraph.*]



Definition. A graph H is a **minor** of a graph G if H can be obtained from G by a sequence of edge deletions and/or edge contractions. [*“Minor” suggests smaller: H is smaller than G .*]

Note. Any subgraph of G is also a minor of G .

Modifications of Graphs

Definition. A **subdivision** of an edge e is the replacement of e by a path of length *at least* two. [*Like adding vertices in the middle of e .*]

Definition. A **subdivision** of a graph H is the result of zero or more sequential subdivisions of edges of H .

Note. If G is a subdivision of H , then G is at least as large as H .

Note. If G is a subdivision of H , then H is a minor of G .
(Contract any edges that had been subdivided!)

Note. The converse is not necessarily true.

Kuratowski's Theorem

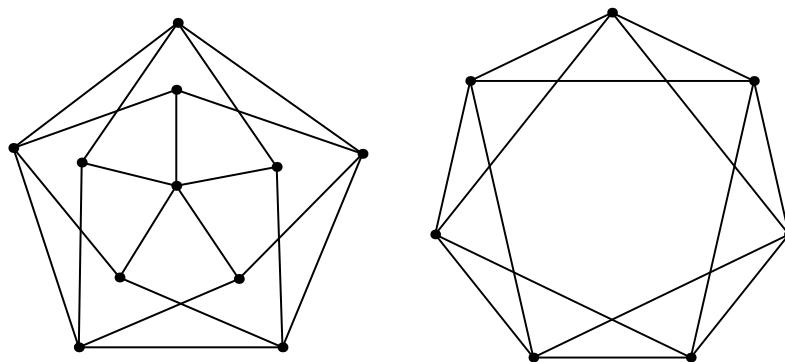
Theorem. Let H be a subgraph of G . If H is nonplanar, then G is nonplanar.

Theorem. Let G be a subdivision of H . If H is nonplanar, then G is nonplanar.

Corollary. If G contains a subdivision of a nonplanar graph, then G is nonplanar.

Theorem. (Kuratowski, 1930) A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Theorem. (Kuratowski variant) A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G .



Kuratowski's Theorem

- ▶ To prove that a graph G is planar, find a planar embedding of G .
- ▶ To prove that a graph G is non-planar, **(a)** Use $q \leq 3p - 6$, or **(b)** find a subgraph of G that is isomorphic to a subdivision of K_5 or $K_{3,3}$, or **(c)** successively delete and contract edges of G to show that K_5 or $K_{3,3}$ is a minor of G .
- ▶ Practice on the Petersen graph. (Here, have some copies!)

