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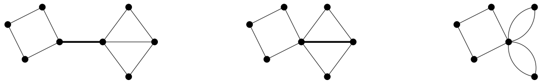
G/e (G contract e): If $e = vw$, coalesce v and w into a super-vertex adjacent to all neighbors of v and w. [*This may produce a multigraph.*]



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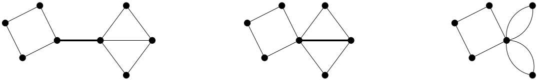

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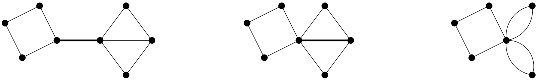

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Note. The converse is not necessarily true.

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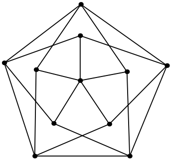

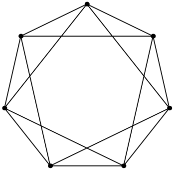

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- Practice on the Petersen graph. (Here, have some copies!)