MATH 636, Fall 2014
Practice Problems
in preparation for Exam 2 on Thursday, December 3, 2015

The exam covers:

- *Combinatorics: A Guided Tour*, Sections 2.4, 3.3–3.6 (not including EGFs), 4.1, 4.4.
- Additional topics that are included in the course notes, including generating function manipulations and applications, products and compositions of generating functions, compositions of an integer, standard Young tableaux, Catalan numbers, combinatorial statistics, and q-analogs.

Below are some questions that practice concepts from the class.

- Book exercises: 2.4.6, **2.4.8**, 3.3.5, 3.3.9 (also solve using convolution), 3.4.4, 3.5.1a,e, 3.6.2, **3.6.6** (Also use composition of generating functions), 4.1.13, 4.4.1, 4.4.3, **4.4.9**, (Bolded exercises are especially encouraged.) *Challenge question:* 3.4.11
- Book questions (in the chapters): 123, 172, 174

Q1. Suppose you have an unlimited supply of black building blocks of height 1 and an unlimited supply of red, orange, yellow, green, blue, and purple building blocks of height 2. How many ways are there to build a tower of height n?

Q2. We can make a four-sided die using a tetrahedron. If the sides of the die are 1, 2, 3, and 4, then there is a particular distribution of possible sums between 2 and 8. Determine a relabeling of the two dice which gives the exact same distribution of sums. How many possibilities are there?

Q3. The Catalan numbers c_n satisfy the generating function

$$C(x) = \sum_{n\geq 0} c_n x^n = \frac{1-\sqrt{1-4x}}{2}$$

Consider an integer sequence $\{d_n\}_{n\geq 0}$ that has as its generating function

$$D(x) = \sum_{n\geq 0} d_n x^n = 1 - 5\sqrt{1-4x}.$$

Determine a formula for d_n written as a function of Catalan numbers.

Q4. Investigate and conjecture a formula for the number of standard Young tableaux of shape $\lambda : 2n = n + n$ for all positive n. For example, when $n = 3$, two valid tableaux are $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}$. Prove your conjecture by finding a bijection with a combinatorial interpretation we have seen previously.

[*Hint: You will likely use the fact that a standard Young tableau of shape* (n, n) *is completely defined by its first row.*]

Q5. Show that the number of Young diagrams that fit inside an $m \times n$ rectangle is $\binom{m+n}{m}$. [*Hint: look for a bijection with walks in a lattice.*]