Course Notes

Combinatorics, Fall 2015

Queens College, Math 636

Prof. Christopher Hanusa

http://qc.edu/~chanusa/courses/636/15/

Reference List

The following are books that I recommend to complement this course. They are *on reserve* in the library.

Benjamin and Quinn. *Proofs that really count.*
Bóna. *A walk through combinatorics.*
Brualdi. *Introductory combinatorics.*
Graham, Knuth, and Patashnik. *Concrete mathematics.*
Mazur. *Combinatorics: A guided tour*
van Lint and Wilson. *A course in combinatorics.*

What is combinatorics?

In this class: Learn how to count . . .

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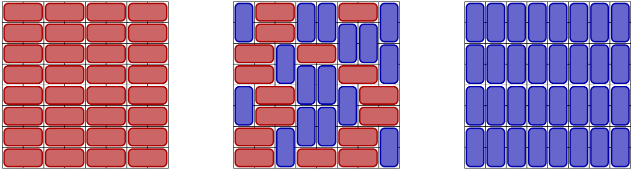

A **domino tiling** is a placement of dominoes on a region, where

- Each domino covers two squares.
- The dominoes cover the whole region and do not overlap.

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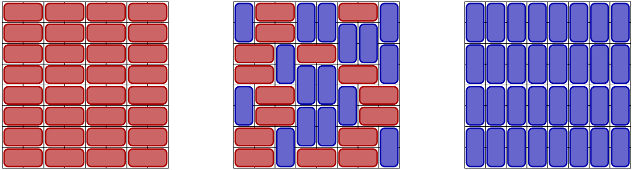

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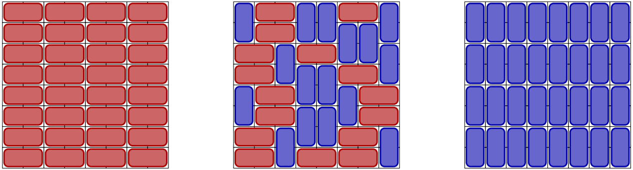

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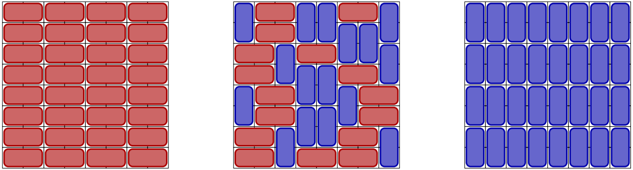

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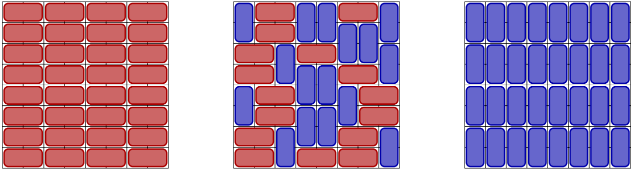

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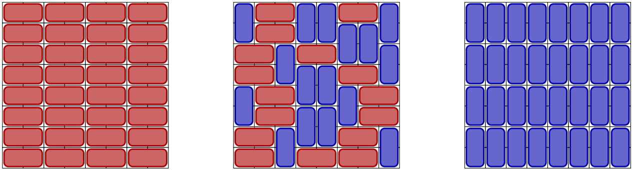

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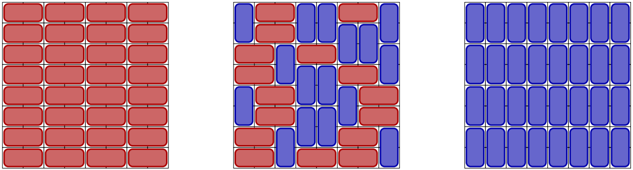

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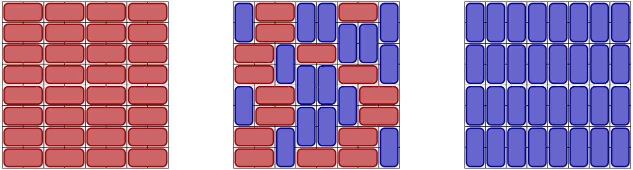

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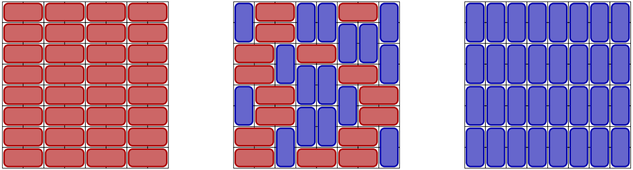

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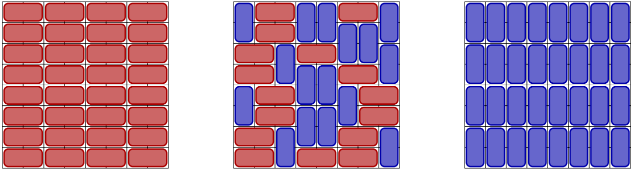

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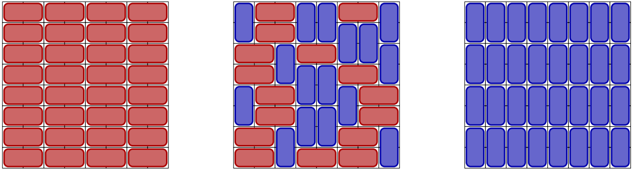

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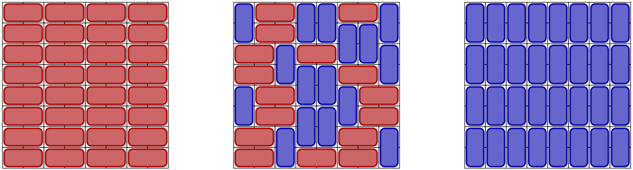

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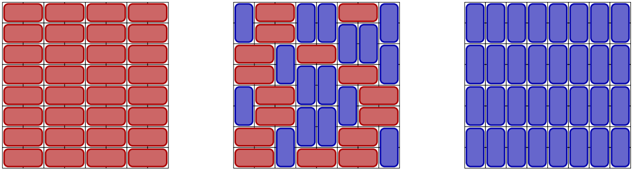

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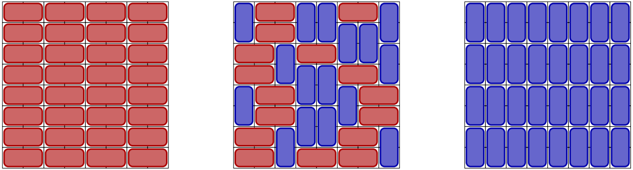

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We have the **answer**! What does it mean?

Domino tilings

How to determine the "answer"?

- Convert the chessboard into a combinatorial structure (a graph).
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Question: How many domino tilings are there of an $m \times n$ board?

Answer: If m and n are both even, then we have the **formula** (!):

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right).$$

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Combinatorial questions

Given some discrete objects, what properties and structures do they have?

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Mastering "Combinatorics" means internalizing many different techniques and strategies to know the best way to approach any counting question. We will develop **our toolbox.**

Uses a different kind of reasoning than in other math classes.

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All homeworks online. Email me by Tuesday, 1st hwk due Thursday.

Numbers are everywhere

Arrange yourselves into groups of four people,
With people you **don't know**.

- Introduce yourself. (your name, where you are from)
- What brought you to this class?
- Fill out **the front of** your notecard:
  - Write your name. (Stylize if you wish.)
  - Write some words about how I might remember you & your name.
  - *Draw* something (anything!) in the remaining space.
- Exchange contact information. (phone / email / other)
- *Small talk suggestion:* What kept you busy this summer?

Four Counting Questions (p. 2)

Here are four counting questions.

Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Q3. How many Pick-6 lottery tickets are there? (Choose six numbers between 1–40.)

Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Group discussion: Use your powers of estimation to order these from smallest to largest.

_____ _____ _____ _____

Counting words

Definition: A **list** or **word** is an ordered sequence of objects.

Definition: A k-**list** or k-**word** is a list of length k.

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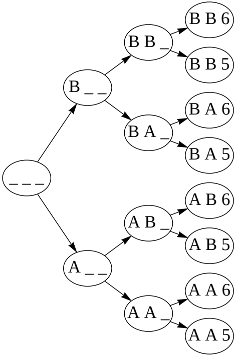

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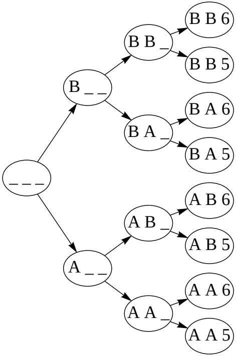

Alternatively: Notice two *independent* choices for each character. Multiply $2 \cdot 2 \cdot 2 = 8$.

The Product Principle

This illustrates:

The product principle: When counting lists $(l_1, l_2, \ldots, l_k)$,

IF there are c_1 choices for entry l_1, each leading to a different list,

AND IF there are c_i choices for entry l_i,
no matter the choices made for l_1 **through** l_{i-1},
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THEN there are $c_1 c_2 \cdots c_k$ such lists.

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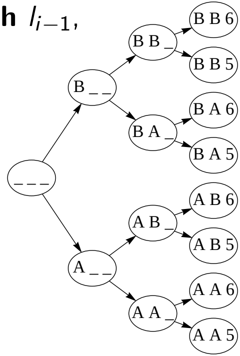

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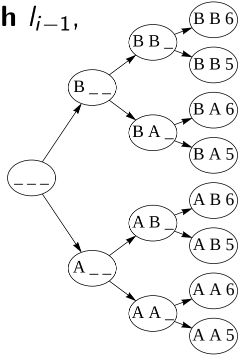

Caution: The product principle seems simple, but we must be careful when we use it.

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(=218,340,105,584,896)

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In general, the number of words of length k that can be made from an alphabet of length n and where repetition is allowed is n^k

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We can label the subsets by whether or not they contain s_i.

For example, for $n = 3$, we label the subsets $\left\{ \begin{matrix} 000, 100, 010, 110, \\ 001, 101, 011, 111 \end{matrix} \right\}$

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Answer: The number of choices for each lineup spot are:

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____ ____ ____ ____ ____ ____ ____ ____ ____.

Multiplying gives that the number of lineups is ____ $= 362{,}880$.

Definition: A **permutation** of an n-set S is an (ordered) list of **all** elements of S. There are $n!$ such permutations.

Definition: A k-**permutation** of an n-set S is an (ordered) list of k distinct elements of S. How many are there?

- "Permutation" always refers to a list without repetition.

Lists WITHOUT repetition

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In general, the number of words of length k that can be made from an alphabet of length n and where repetition is NOT allowed is $(n)_k$.

- That is, the number of k-permutations of an n-set is $(n)_k$.
- Special case: For n-permutations of an n-set: $n!$.

Notation

Some quantities appear frequently, so we use shorthand notation:

- $[n] := \{1, 2, \ldots, n\}$
- $2^S :=$ set of all subsets of S
- $n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$
- $(n)_k := n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \dfrac{n!}{(n-k)!}$

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- $\dbinom{n}{k} := \dfrac{n!}{k!(n-k)!} = \dfrac{(n)_k}{k!}$
- $\left(\!\!\dbinom{n}{k}\!\!\right) := \dbinom{k+n-1}{k}$

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Answer: $\binom{40}{6} = 3{,}838{,}380$.

- $\binom{n}{k}$ is called a **binomial coefficient**.
- Alternate phrasing: How many k-subsets of an n-set are there?
- The individual objects we are counting are unordered.
  They are <u>subsets</u>, not lists.

A formula for $\binom{n}{k}$

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Since we counted the same quantity twice, they must be equal!

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Think Write Pair Share: Enumerate **all** multisubsets of [3].

[In other words, *list them all* or *completely describe the list*.]

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Answer:

How would you describe a k-multisubset of $[n]$?

Stars and Bars

Question: How many k-multisets can be made from an n-set?

— is the same as —

Question: How many ways are there to place k indistinguishable balls into n distinguishable bins?

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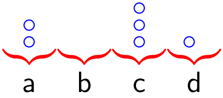

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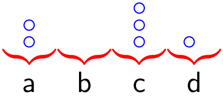

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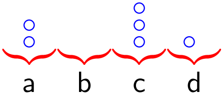

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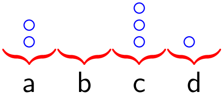

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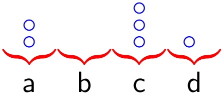

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Answering Q1–Q4

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Answer: $\left(\!\!\binom{\quad}{\quad}\!\!\right) = \binom{\quad}{\quad} = 7{,}898{,}654{,}920.$

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Answer: $\left(\!\!\binom{\ }{\ }\!\!\right) = \binom{\ }{\ } = 7{,}898{,}654{,}920.$

Correct order:

Q2.	Order 9 baseball players $(9!)$	362,880
Q3.	Pick-6; numbers 1–40 $\binom{40}{6}$	3,838,380
Q4.	12 donuts from 30 $\left(\!\!\binom{30}{12}\!\!\right)$	7,898,654,920
Q1.	8-character passwords (62^8)	218,340,105,584,896

Summary

	order matters (choose a list)	order doesn't matter (choose a set)
repetition allowed		
repetition not allowed		

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