Counting integral solutions

Question: How many non-negative integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$?

- Give some examples of solutions.
- Characterize what solutions look like.
- A combinatorial object with a similar flavor is:

In general, the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = k$ is ________.

Question: How many **positive** integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$, where $x_4 \geq 3$?

The sum principle

Often it makes sense to break down your counting problem into smaller, disjoint, and easier-to-count sub-problems.

Example. How many integers from 1 to 999999 are palindromes?

Answer: Condition on how many digits.

- Length 1:
- Length 2:
- Length 3:
- Length 4:
- Length 5,6:
- **Total**:

★ Every palindrome between 1 and 999999 is counted once.

This illustrates the **sum principle**:

Suppose the objects to be counted can be broken into k disjoint and exhaustive cases. If there are n_j objects in case j, then there are $n_1 + n_2 + \cdots + n_k$ objects in all.

Counting pitfalls

When counting, there are two common pitfalls:

- Undercounting
  - Often, forgetting cases when applying the sum principle.
  - **Ask:** Did I miss something?
- Overcounting
  - Often, misapplying the product principle.
  - **Ask:** Do cases need to be counted in different ways?
  - **Ask:** Does the same object appear in multiple ways?

Common example: A deck of cards.

There are four suits: Diamond $\diamondsuit$, Heart $\heartsuit$, Club $\clubsuit$, Spade $\spadesuit$.
Each has 13 cards: Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.

Example. Suppose you are dealt two diamonds between 2 and 10.
In how many ways can the product be even?

Overcounting

Example. In Blackjack you are dealt 2 cards: 1 face-up, 1 face-down. In how many ways can the face-down card be an Ace and the face-up card be a Heart ♡?

Answer: There are __ aces, so there are ___ choices for the down card. There are __ hearts, so there are ____ choices for the up card. By the product principle, there are 52 ways in all.

Except:

Remember to ask: Do cases need to be counted in different ways?

Overcounting

Example. How many 4-lists taken from [9] have at least one pair of adjacent elements equal?

Examples: 1114, 1229, 5555 Non-examples: 1231, 9898.

Strategy:

1. Choose where the adjacent equal elements are. (___ ways)
2. Choose which number they are. (___ ways)
3. Choose the numbers for the remaining elements. (___ ways)

By the product principle, there are ________ ways in all.

Except:

Remember to ask: Does the same object appear in multiple ways?

Counting the complement

Q1: How many 4-lists taken from $[9]$ have **at least one** pair of adjacent elements equal?

—Compare this to—

Q2: How many 4-lists taken from $[9]$ have **no** pairs of adjacent elements equal?

What can we say about:

Q1: **Q2:** **Together:**

Q3:

Strategy: It is sometimes easier to **count the complement**.

Answer to Q3:

Answer to Q2:

Answer to Q1:

Poker hands

Example. When playing five-card poker, what is the probability that you are dealt a full house?

[*Three cards of one type and two cards of another type.*] 5 5 5 K K

Game plan:

- Count the total number of hands.
- Count the number of possible full houses. **# of ways**
  - Choose the denomination of the three-of-a-kind.
  - Choose which three suits they are in.
  - Choose the denomination of the pair.
  - Choose which two suits they are in.
  - Apply the multiplication principle. **Total:**
- Divide to find the probability.

Pascal's triangle

Pascal's identity gives us the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
With initial conditions we can calculate $\binom{n}{k}$ for all n and k.
$\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all n.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1							1

Seq's in Pascal's triangle:

- $1, 2, 3, 4, 5, \ldots$ $\binom{n}{1}$
  $(a_n = n)$ A000027
- $1, 3, 6, 10, 15, \ldots$ $\binom{n}{2}$
  triangular A000217
- $1, 4, 10, 20, 35, \ldots$ $\binom{n}{3}$
  tetrahedral A000292
- $1, 2, 6, 20, 70, \ldots$ $\binom{2n}{n}$
  centr. binom. A000984

Online Encyclopedia of Integer Sequences:
`http://oeis.org/`

Binomial Theorem

Theorem 2.2.2. Let n be a positive integer. For all x and y,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n.$$

Rewrite in summation notation!

Determine the generic term $[\binom{n}{k}x \quad y \quad]$ and the bounds on k

$$(x+y)^n = \sum$$

- The entries of Pascal's triangle are the coefficients of terms in the expansion of $(x+y)^n$.

Proof. In the expansion of $(x+y)(x+y)\cdots(x+y)$, in how many ways can a term have the form $x^{n-k}y^k$?

From the n factors $(x+y)$, you must choose a "y" exactly k times. Therefore, $\binom{n}{k}$ ways. We recover the desired equation. □